Mathematical properties of decimal counting boards

Joint Mathematics Meetings, San Antonio, TX
Session on Research in Number Theory 1
Room 210 B Convention Center
1:45-2:00 PM
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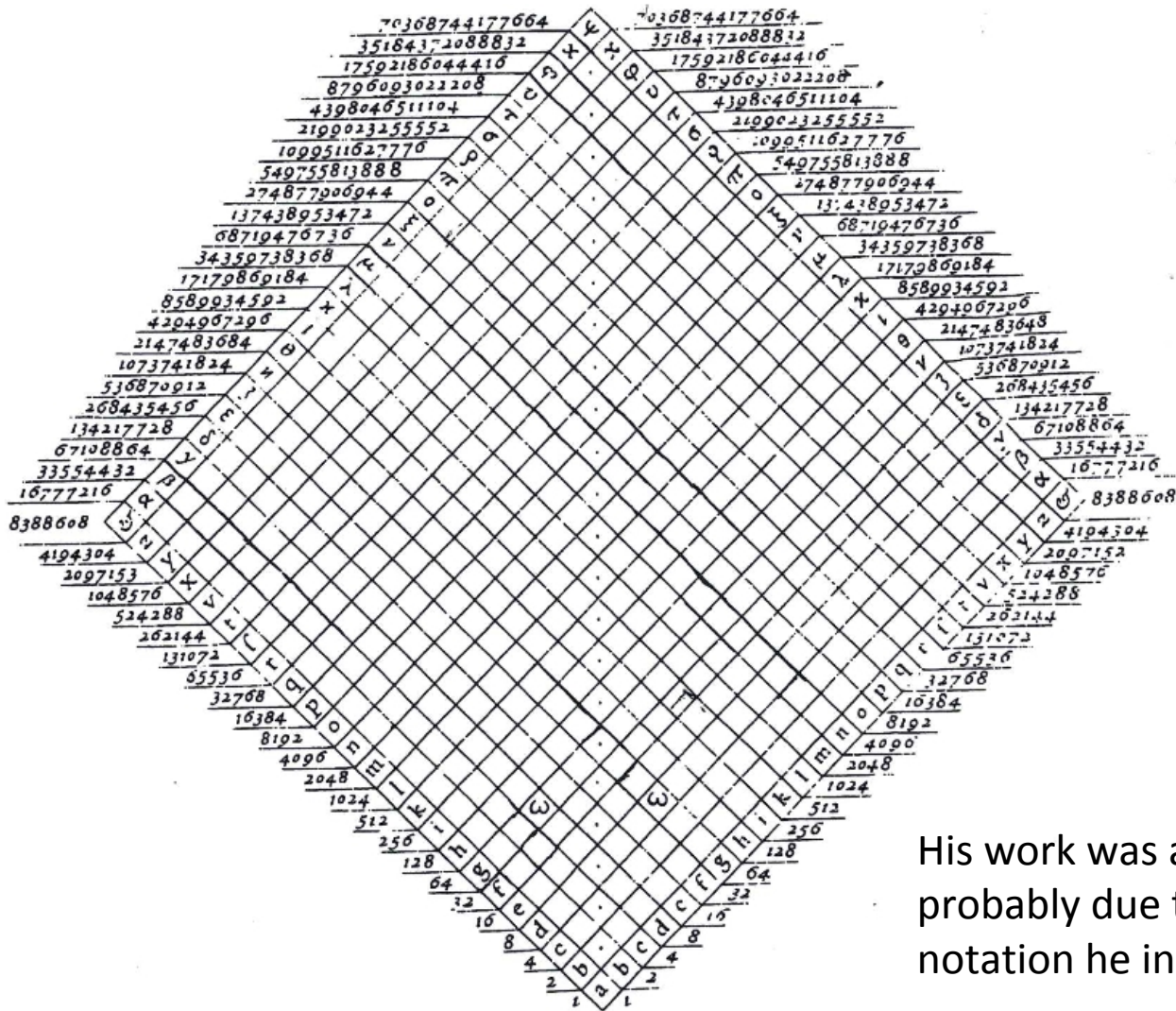New Mexico State Univ.
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In 1617, the year of his death, and three years after his book on logarithms appeared, John Napier published his *Rabdology*.

In it, he introduced a counting board consisting of a square grid whose locations were labeled with numbers where both columns and rows formed geometric progressions with the quotient 2.

The board supported very efficient algorithms.

His work was almost completely ignored, probably due to the rather unusual notation he invented and used.

We adapted his idea to better match modern decimal notation, by using geometric progressions with quotient 2 in columns, and quotient 5 in rows. Also we use two-color tokens to represent both positive numbers (white) and negative numbers (red). We call such boards "decimal boards".

Here, 18.84 is shown as the sum 20 + -2 + .8 + .04 .

We have experimented with using the boards to explain arithmetic operations and corresponding algorithms to both students and teachers in elementary and middle grades. So far the results look very promising.

In practice, the size of the boards that are used depends both on the type of tasks involved and the level of the students.
But it is convenient to view a board as an infinite grid of squares on which any finite decimal can be represented.

The basic question that is asked in each grade is, what configurations of tokens represent a given number?
 In mathematical terms, this corresponds to presenting a finite decimal as a sum of powers of $2^i * 5^j$ and their negations, where i and j are integers.

For example, the number three can be represented by 2 tokens in 8 different ways,

2 + 1,
4 − 1,
5 − 2,
8 − 5,
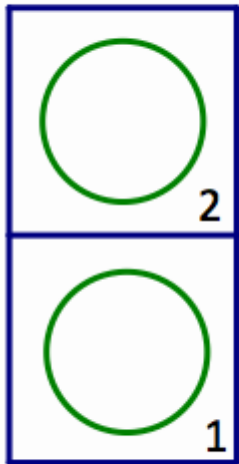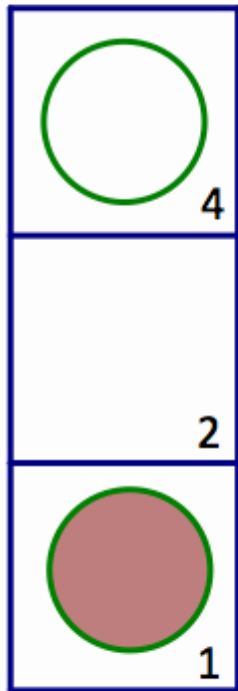128 − 125,
2.5 + .5,
3.2 - .2, and
3.125 - .125
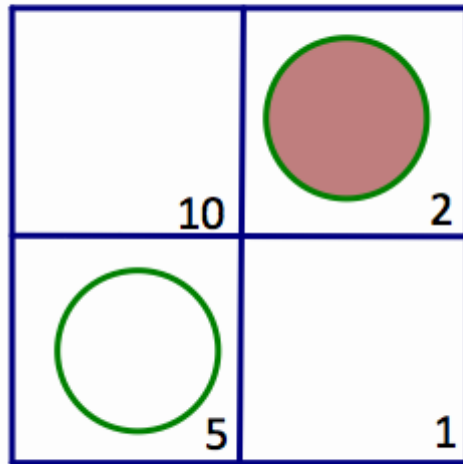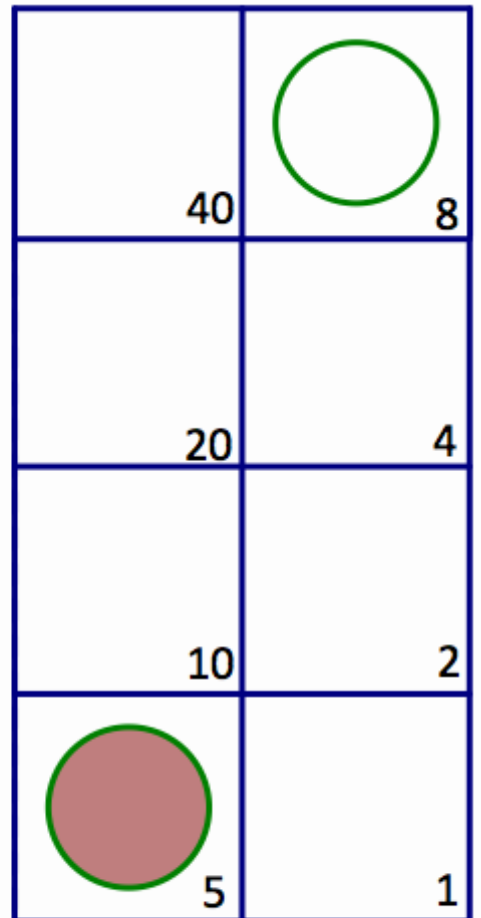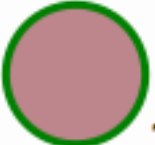
2+1   4-1   5-2   8-5

16000	3200	640	○ 128
8000	1600	320	64
4000	800	160	32
2000	400	80	16
1000	200	40	8
500	100	20	4
250	50	10	2
● 125	25	5	1
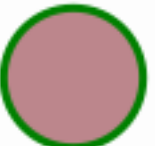
128-125

5	1
○ 2.5	○ .5

2.5 + .5

16	◯ 3.2
8	1.6
4	.8
2	.4
1	● .2

3.2-.2

25	5	1
12.5	2.5	.5
6.25	1.25	.25
◯ 3.125	.625	● .125

3.125-.125

A general framework to answer such questions

Consider the set S containing the two primes 2 and 5, S = {2, 5}.
The set of S-units consists of all numbers $2^i 5^j$ and $-2^i 5^j$, where i and j are integers.

An S-unit equation is an equation, $x_1 + x_2 + \dots + x_k = d$, where d is a finite decimal, and its solutions are S-units.

Any representation of a finite decimal d by k tokens corresponds to a solution of such an equation. And any solution to such equation describes a configuration of tokens representing d.

A brief survey of the results about S-equations, which provides a very complete set of references, can be found in
Rational Number Theory in the 20th Century by Wladyslaw Narkiewicz, Springer 2012, pages 255-257.

For example:
There is a general theorem from which it follows that there is an algorithm that checks whether or not the set of 8 representations of the number 3 given above is complete.
But no such algorithm has been implemented, and we also don't know whether it would give us an answer in a reasonable time.
So it is still possible that there are some other ways of representing the number 3 on a decimal board.

Remark: There are some specific results concerning the set S = {2, 3}, but we didn't find any about S = {2, 5}.

A question about boards of a given size

Students always use a board of a specific size, and usually its size is rather small. So the question regarding how a given number is represented on a particular board can often be solved by some "ad hoc" method.

But even for rather small boards, an exhaustive search through all cases is not feasible. And this leads to the following problem.

Given a board with n rows and m columns, and a decimal d, what is the complexity of the algorithm that returns all representations (if any) of d, on this board?
At present we don't know whether there is a polynomial-time algorithm that does this task, but we think that it is rather unlikely.

Questions about approximations of real numbers

On an infinite decimal board, only finite decimals have exact representations, but any real number can be approximated with any degree of accuracy by *one* token. This is so because for any real number r and $\varepsilon > 0$, there are integers i and j, such that $|r - 2^i 5^j| < \varepsilon$.

But at the present time we do not know anything about which real numbers can be represented on "small" boards with a "few" tokens.

Three questions

1. Given a number, how many ways can we represent it on an infinite board?

2. Given a board of a specific size, how many ways can we represent a given number?

3. How well can we represent a given real number on a finite board?

We would be happy if someone would provide us with answers to the questions we have asked in this talk!

More about this work can be found on our website,
http://www.math.nmsu.edu/~breakingaway/

Click on Notes, and you will find several presentations dealing with decimal boards.

Titles of presentations on our website,
http://www.math.nmsu.edu/~breakingaway/:

Number words and arithmetic: A story of numbers

Explaining arithmetic algorithms

Arithmetic algorithms taught in schools

Was John Napier the first modern computer scientist?

Two models of learning the concept of whole numbers

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Thank you!