Projects for Liberal Arts Majors Explaining the Concept of Limit in Probability Problems Simulated on the TI-84 Calculator

MAA Session on Projects, Demonstrations, and Activities that Engage Liberal Arts Mathematics Students, I

JMM San Diego

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Outline

1. Our liberal arts course, Math Appreciation
2. Some projects used in the course
3. Questions asked of students, and their responses
4. Mathematical definitions of some concepts
5. Final remarks

1 . Our liberal arts course, Math Appreciation

At New Mexico State University, Math Appreciation (Math 210) is taken mostly by students who just want to fulfill minimal math requirements.

Students have limited math backgrounds, and instructors are free to design materials for their sections of the course any way they want.

In our section in Fall 2012 we gave students a sequence of hands-on and virtual tasks, simulated on the TI-84 calculator.

Students were told before they enrolled that there would be no textbook, but there would be daily handouts, and they would be required to have a TI-83/84 graphing calculator.

2 . Some projects used in the course

One group of tasks used in the class dealt with theoretical probability distributions and distributions observed from samples.

This is a difficult topic because convergence from sample distributions to a theoretical distribution is so slow that students often come to the conclusion that “A theoretical distribution is something that *should* happen, but it never happens in the real world.”

In the class, students first experimented with sixand twenty-sided dice.

And then they entered into their calculators and ran simulations that displayed the distributions from various sample sizes, both numerically and graphically.

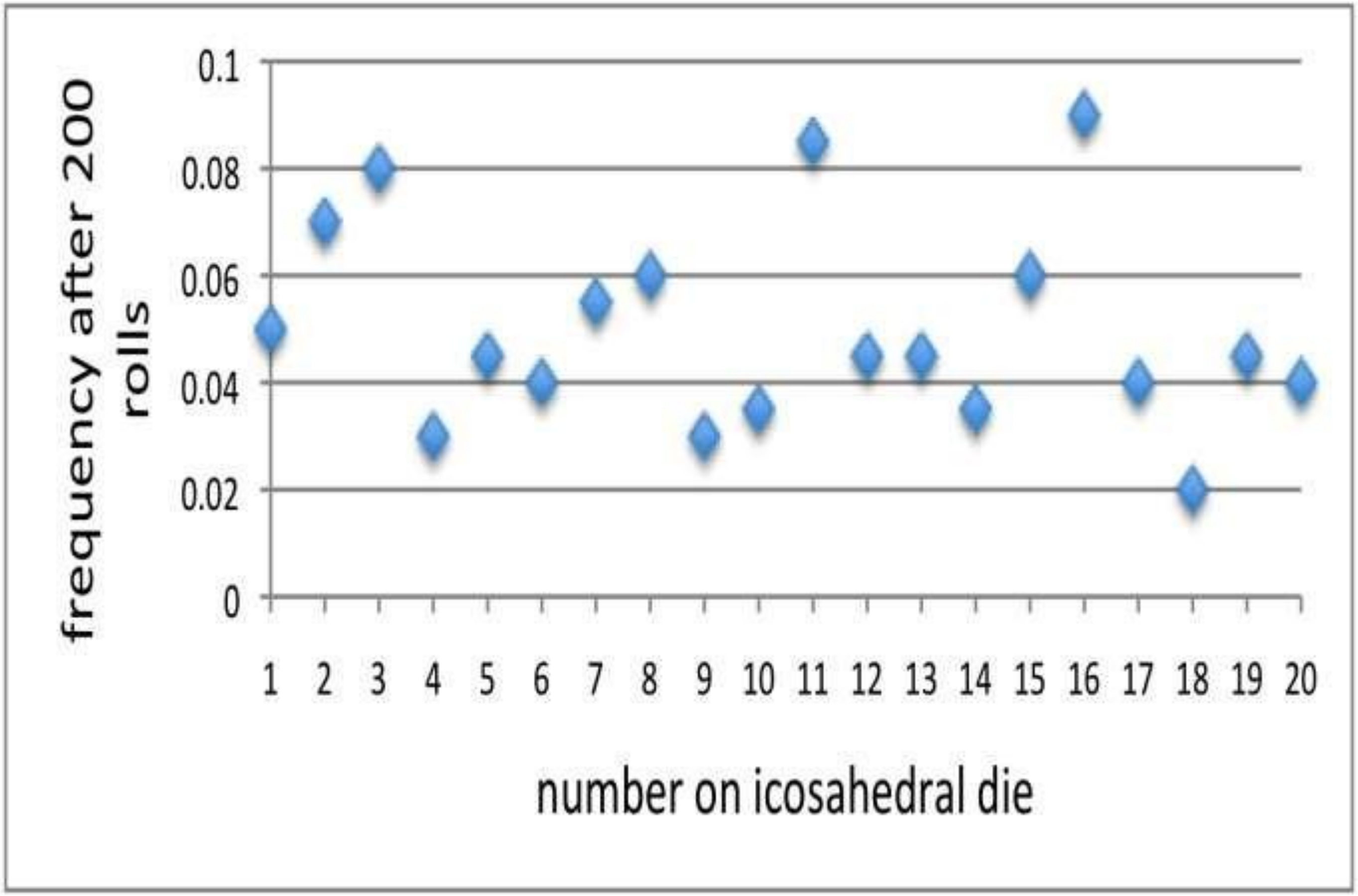
(In order to match a theoretical distribution within one percentage point, a twenty-sided die has to be tossed several thousand times.)

The students were given the code for a program that simulates rolling one twenty-sided die.

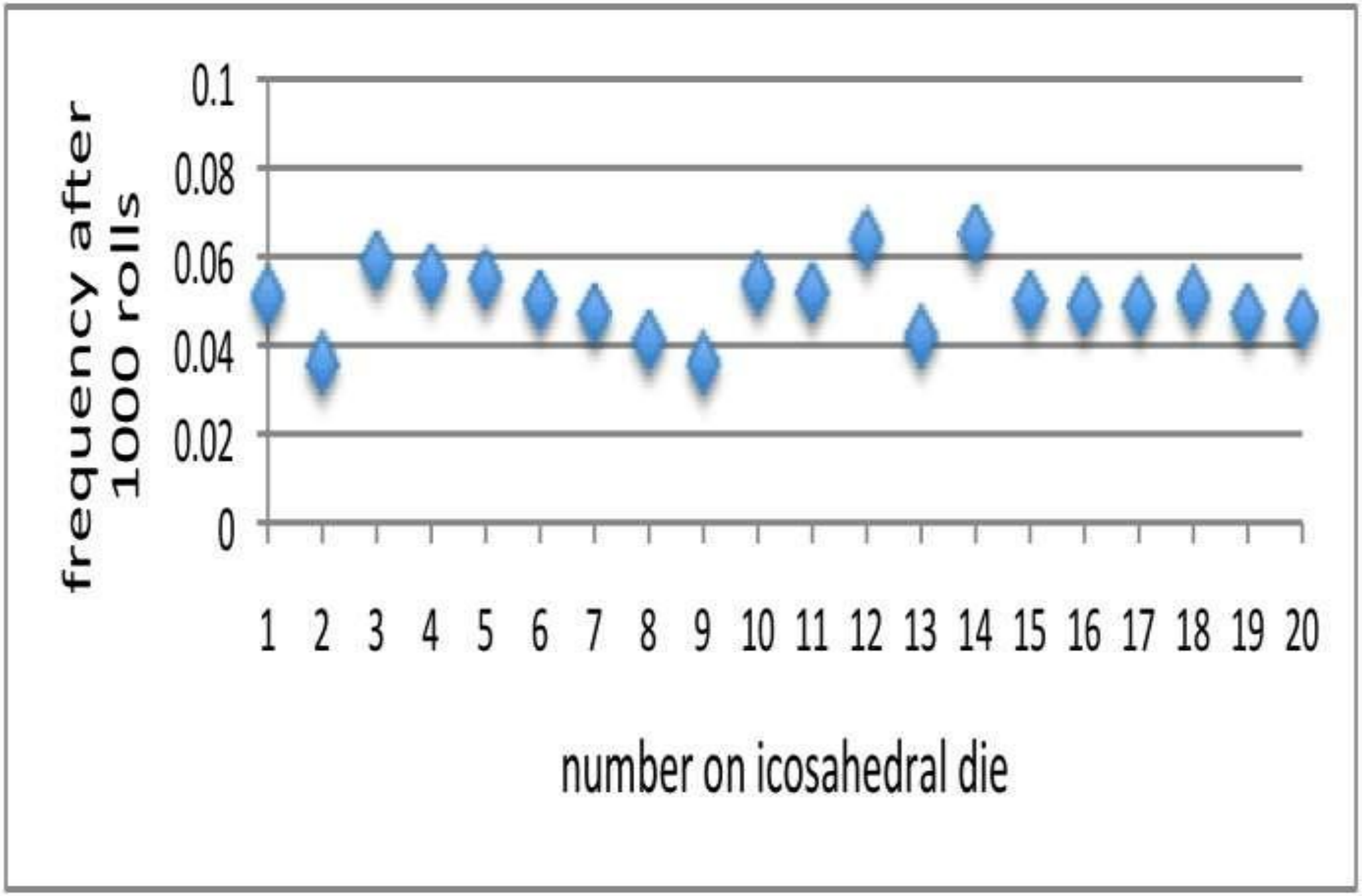
Students entered the program into their calculators, set the calculator window and other parameters, and ran it.

After they entered the number of sides of the die, the program created a sequence of ten consecutive graphs, each representing the roll of 20 dice.

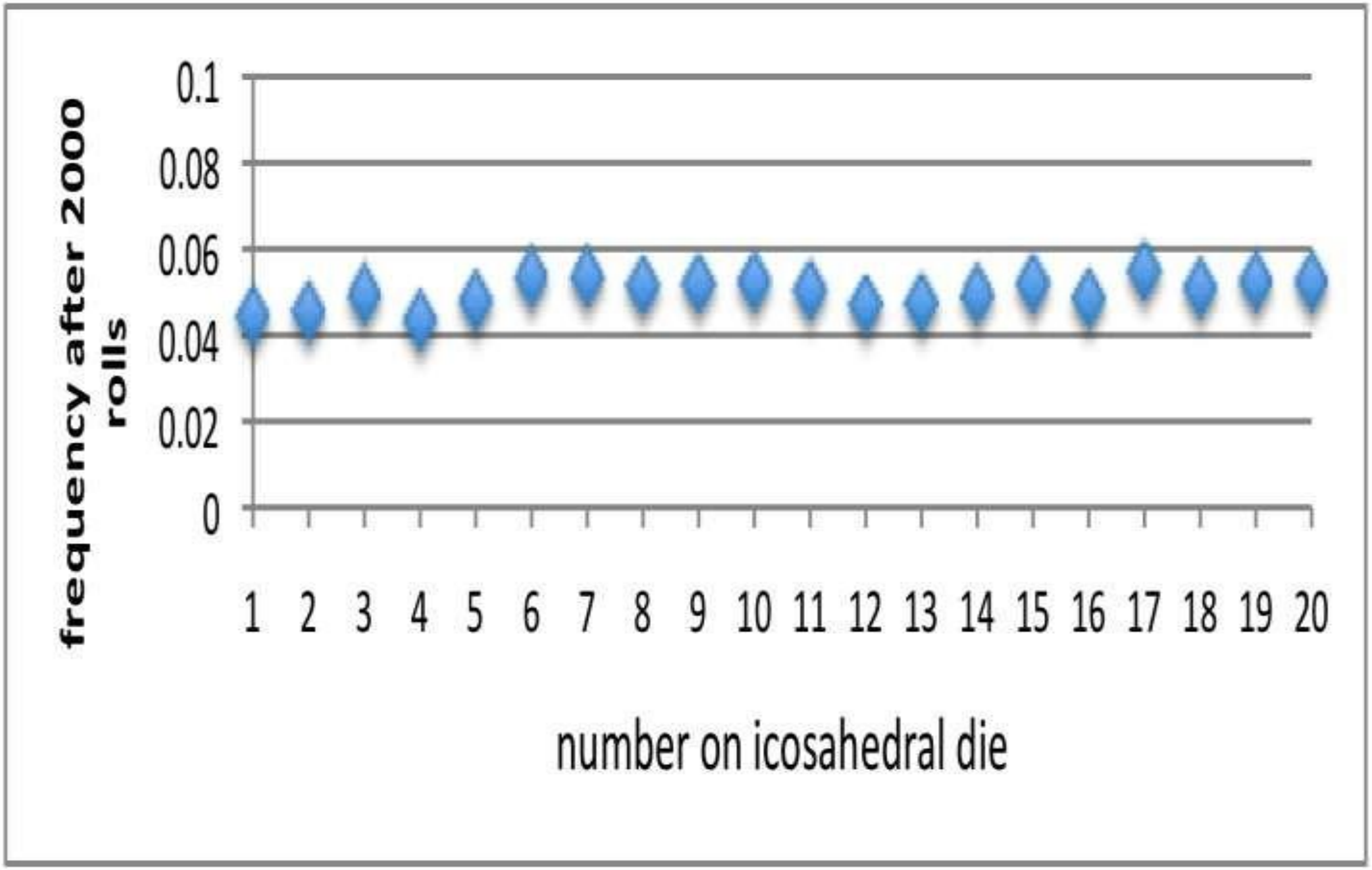
(The theoretical distribution is of course .05 for each outcome.) Here is an example of the distribution after 200 rolls:



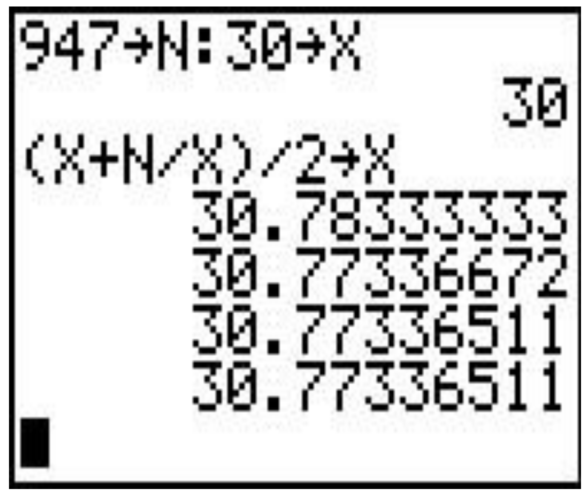
Here is a distribution after 1000 rolls:



And an example after 2000 rolls:

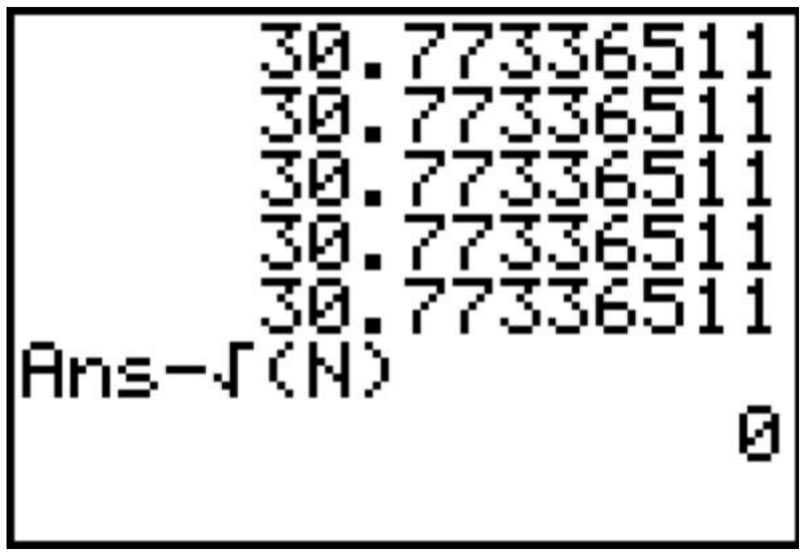


These activities were followed by looking at several numerical sequences and series, some that were convergent, either fast or slowly, and some that were divergent. Examples included computing a square root by Newton's method, as below:



The square root of 947, on the calculator, is 30.77336511.

And to check that this is correct, we compare the answer to the square root of 947:



3 . Questions asked of students, and their

responses

We wanted to know what concepts of convergence and limit students formed, so we gave them an anonymous questionnaire near the end of the semester, and we included some questions about limits on the final exam.

Anonymous questionnaire

This is not a test. We are not looking to see if your answers are correct or not, but we plan to use your answers to prepare better teaching materials. Thank you.

Imagine this situation. A high school, or even middle school, student asks you these questions:

What does it mean that a sequence of numbers is convergent?

What does it mean that a sequence of numbers is divergent?

What is a limit of a sequence of numbers?

How would you answer these three questions? Would you use examples? Would you draw a picture? Would you use a calculator? Please write down how you would answer the young students' questions. Your explanations:

Any additional comments and explanations are appreciated. (Please use the back if needed.)

Questions on the final exam

8. Give an example of a divergent sequence.
9. Give an example of a convergent sequence.
12. What is the limit of the series $^1/_2 + {}^1/_4 + {}^1/_8 + {}^1/_{16} + \ldots$? How do you know?

Answering questions anonymously is important because students tell what they think, instead of trying to give a “correct” answer.

An example of a student’s response

Your explanation:

I would tell the student to look the answers up online, and watch some YouTube videos. This is how I learned math in high school.

Any additional comments…

I honestly do not remember anything about convergence or divergence.

The concept of a convergent sequence

Answers of students who seemed to understand the concept fell into four categories:

- The numbers of the sequence come closer and closer together.
- The numbers of the sequence will eventually meet.
- The numbers of the sequence approach some limit. • The numbers of the sequence stop changing.

In answering these questions, no student used the term "infinite sequence". Only one student gave a "plausible" explanation of a limit:

"The limit of a sequence of numbers is essentially where the sequence stops."

During the final exam, which was open book, open notes, open internet, most students gave very standard examples of convergent sequences, such as 1, $^1/_2$, $^1/_3$, $^1/_4$, …□ 0

But some students presented finite sequences as examples of convergent sequences, and one student even gave this explanation:

1 /4, 2/4, 3/4, 4/4 = 1, it converges because it stops at 1.

In class, students handled sequences of lengths up to 100 during hands-on activities involving dice.

And when running simulations, they handled sequences of lengths up to 10,000, which were entered as algebraic formulas or programs into their calculators.

They also talked about (infinite) sequences that “go on forever”.

4 . Mathematical definitions of some concepts

The distinction between finite and infinite sequences was never an important issue in the class.

So the following definitions of a sequence of real numbers, convergence, and limit are consistent with students' experiences and with the informal concepts they formed: Definitions

A sequence of real numbers s is a function from a subset D of integers, where D has a smallest element, into the set of real numbers. (The set D is the domain of sequence s.)

This definition is more general than the one that is commonly used, because it includes *finite* sequences, and it allows negative indices and gaps between consecutive indices. Its advantage is that it includes vectors and lists, and other (finite) sequences that can be processed on calculators.

Of course any sequence s has a unique first element, second element, etc. So the standard notation s_1, s_2, … can be used for any domain D.

To define the remaining concepts, we start with an auxiliary definition:

A tail T(i), where i □ D, of a sequence s, is the set of all s(j), where j □ D, and j ≥ i.

A sequence s is convergent if and only if for every positive integer k there is i □ D such that, for any two numbers u, v □ T(i), $|u – v| < 1/k$.

A real number u is the limit of a sequence s(n), where n □ D, if and only if for every positive integer k there is i □ D such that for any number v □ T(i), $|u – v| < 1/k$.

The relation between convergence and limits is described by this theorem:

A sequence s has a limit if and only if it is convergent.

The distinction between finite and infinite sequences is simple:

A sequence is finite if and only if it has a last element.

These definitions differ from the standard Cauchy definitions only in one aspect. According to the definitions here, any non-empty *finite* sequence is convergent, and its last term is its limit.

The concept of a divergent sequence

Only one student wrote that

A sequence is divergent when it is not convergent.

The answers of most students fell into two categories:

- The numbers of the sequence are moving away from each other.

- The numbers of the sequence form a random pattern.

Examples of a divergent sequence given on the final looked mostly like this:

1 , 2, 3, 4, …

or 1, 2, 3, 4, … □ $^{\infty}$ or ,

2, 3, 4 … limit = ∞ 1

The concept of “infinite number”, ∞, was not used in class, so students must have gotten it from some other sources.

Students were shown divergent sequences such as

0 , 1, 0, 1, 0, 1, …,

but still they formed only very narrow concepts of a divergent sequence.

5 . Final remarks

Teaching mathematical concepts by embedding them in some activities that are meaningful to students, followed by informal explanations, and finalized by mathematical definitions, is a technique that is often used and strongly recommended.

But there is a danger that during the preliminary stages students will form concepts that are inconsistent with the final mathematical definitions. And this can lead to confusion and errors.

We think that in order to avoid inconsistencies, one needs either to prepare activities that exactly match mathematical definitions, or to adjust mathematical definitions to fit the students’ experiences.

We think that the definitions of a sequence, and of convergence and limit, that are shown above fit the experiences of students who work with physical processes or their simulations better than do the standard definitions.