Number Words and Arithmetic: A Story of Numbers

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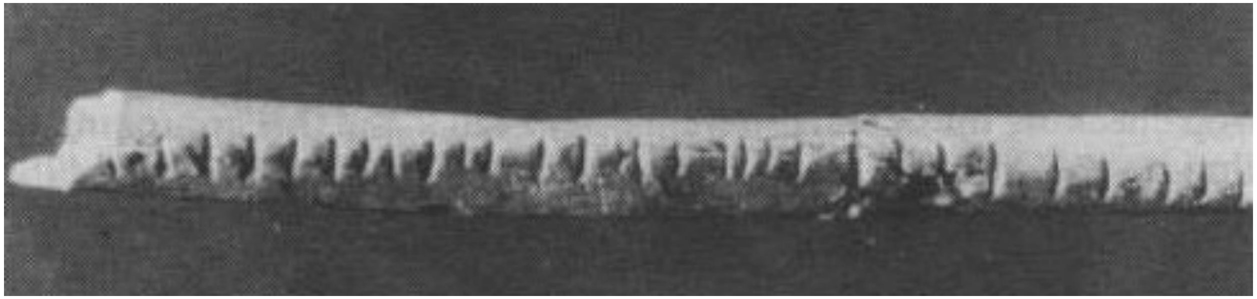
Outline

1. The invention of numbers as suggested by John Leslie (1817)
2. John Napier's original abacus (from *Rabdologiæ,* 1617), and bases other than two
3. The current approach to early learning of arithmetic
4. An alternative approach to introducing numbers

1. The invention of numbers as suggested by John Leslie (1817)

The oldest manifestations of arithmetic are tally marks on Neolithic bones.



Lebombo bone ~35,000 years old

Ishango bone ~ 20,000 years old

But there is no evidence that the tally marks were associated with a concept of number.

It is easy to imagine that a set of tally marks represents a set of objects or events, where each mark represents one element.

Thus the tally | | | | | | may represent a group of consecutive days, one | for each day, without any conception of the number six.

THE

PHILOSOPHY

OF

ARITHMETIC;

EXHIBITING

A PROGRESSIVE VIEW

OF THE

THEORY AND PRACTICE OF CALCULATION,

WITH

AN ENLARGED TABLE OF THE PRODUCTS OF NUMBERS
UNDER *ONE HUNDRED.*

BY

JOHN LESLIE, F. R. S. E.

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY
OF EDINBURGH.

EDINBURGH:

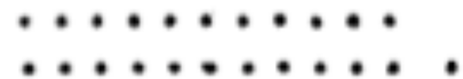
Printed by Abernethy & Walker,
FOR ARCHIBALD CONSTABLE AND COMPANY, EDINBURGH;
AND LONGMAN, HURST, REES, ORME, AND BROWN
LONDON.

1817.

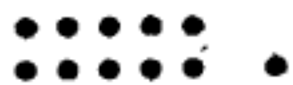
Let's look at an idea presented in John Leslie's *The Philosophy of Arithmetic* (1817) about the invention of numbers.

Let us endeavour to trace the steps by which a child or a savage, prompted by native curiosity, would proceed in classing, for instance, *twenty-three* similar objects.—

1. He might be conceived to arrange them by successive *pairs*. Selecting *twenty-three* of the smallest shells or grains he could find, he might dispose these in *two* rows, containing each *eleven* counters, and *one* over.



Having thus reduced the number to *eleven*, he might subdivide this again, by representing only one of the rows, with shells *twice* as large as before. He would consequently obtain two rows of *five* each, with an excess of *one*.



Leslie, pp. 4-5.

We interpret this to mean that he suggests that the invention of numbers may have started with the discovery than any collection of small objects (for example, pebbles) is either even or odd, but not both.

This meant that attempting to split such collections into two equal parts, "A pebble for me, a pebble for you", ends up either with two equal parts, or with two equal parts and one left over.

The ability to recognize small numbers, one, two, and three, seems to be inborn (e.g., Wynn, 1992; Dehaene, 1997). So the concept of number may have started with the recognition that repeated halving, and creating a record:

even, odd, odd, ..., even,

which ends with a record of either two || or three ||| items, provides a unique description of the total.

The key idea from Leslie:
Processing such records, for example, finding sums and differences, can be done without having names for individual numbers.
Therefore the beginning of arithmetic may have started with the processing of such records and not with creating number names.

Let's see how this might work.
(Taken from our JMM talk in Boston, "Counting", January 2011.)

A person doing this task needs only to record, for each iteration, whether there is a leftover or not, and to record the number of tokens left at the end. It doesn't require any mathematical knowledge or linguistic literacy.

A whole record may look like this:

o	no leftover
|	one leftover
|	one leftover
o	no leftover
|	one leftover
///	three tokens left at the end

Do you see that this record came from partitioning 118 tokens?

3 → 7 → 14 → 29 → 59→ 118

Actual procedure for partitioning 118 tokens

Equipment needed: Three bowls, which we call L, M, and R (Left, Middle, and Right) and material for recording.
To begin, put all tokens in the middle bowl

Move the tokens by taking one or two tokens from the middle container in each hand and putting them simultaneously into the two side containers. You may look at what you are doing, or do it just by touch.

No. of tokens in containers:	L	M	R	Record written horizontally:
1st cycle	0	118	0	
	1	116	1	
	2	114	2	
	…	……	…	
	58	2	58	
	59	0	59	o

After the 1st cycle:

	L	M	R	
2nd cycle	59	59	0	
	60	57	1	
	….	….	….	
	88	1	29	o |
3rd cycle	89	29	0	
	90	27	1	
	….	….	….	
	103	1	14	o | |
4th cycle	104	14	0	
	105	12	1	
	….	….	….	
	111	0	7	o | | o

After the 4th cycle:

	L	M	R	
5th cycle	111	7	0	
	….	….	….	
	114	1	3	o | | o |
6th cycle	115	3	0	
	118	0	0	o | | o | ///

At the beginning of the 6th cycle:

“Computation” before the concept of number

“Computation” would consist of forming a collection of tokens when a record for the collection is given. For example, the record

o | | o | ///

requires two operations:

- doubling the existing collection ($k \rightarrow 2k$), and
- doubling and adding one to the collection ($k \rightarrow 2k + 1$)

The process of “computation”:

	ooo	ooo	ooo ooo	ooo ooo ooo ooo	
		ooo	ooo ooo	ooo ooo ooo ooo	
Modern numer-		o	o o	o oo	
ical computation:	3	7	14	29	…

But you could give someone the number of elements (tokens) above without the concept of number, by doubling and doubling plus one the amount (which is the reverse of the process of forming a record).

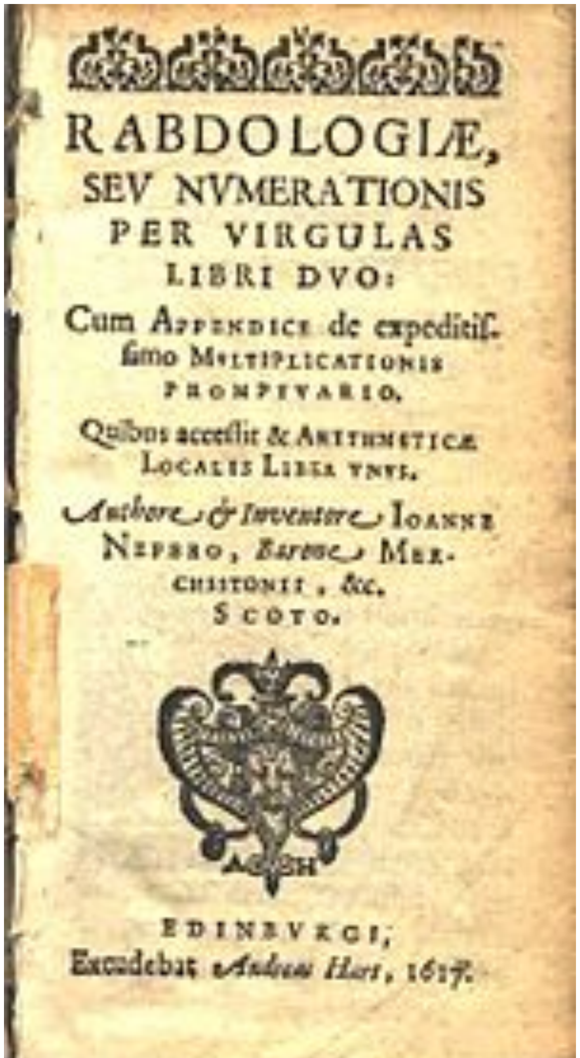
But here our story diverges from Leslie's.

Leslie assumed that the first number system that (the most primitive) people used was the system that we now call "base two". And in this respect he agreed with Levi Conant (1896) and others who followed him, who thought that the size of the base, together with the range of number words (the more the better) was the measure of the mathematical knowledge of a society.

Instead, we think that other bases, such as 10, 20, and 12, could have been used for computation from the very beginning.

Why?

2. John Napier's original abacus (from *Rabdologiæ,* 1617), and bases other than two

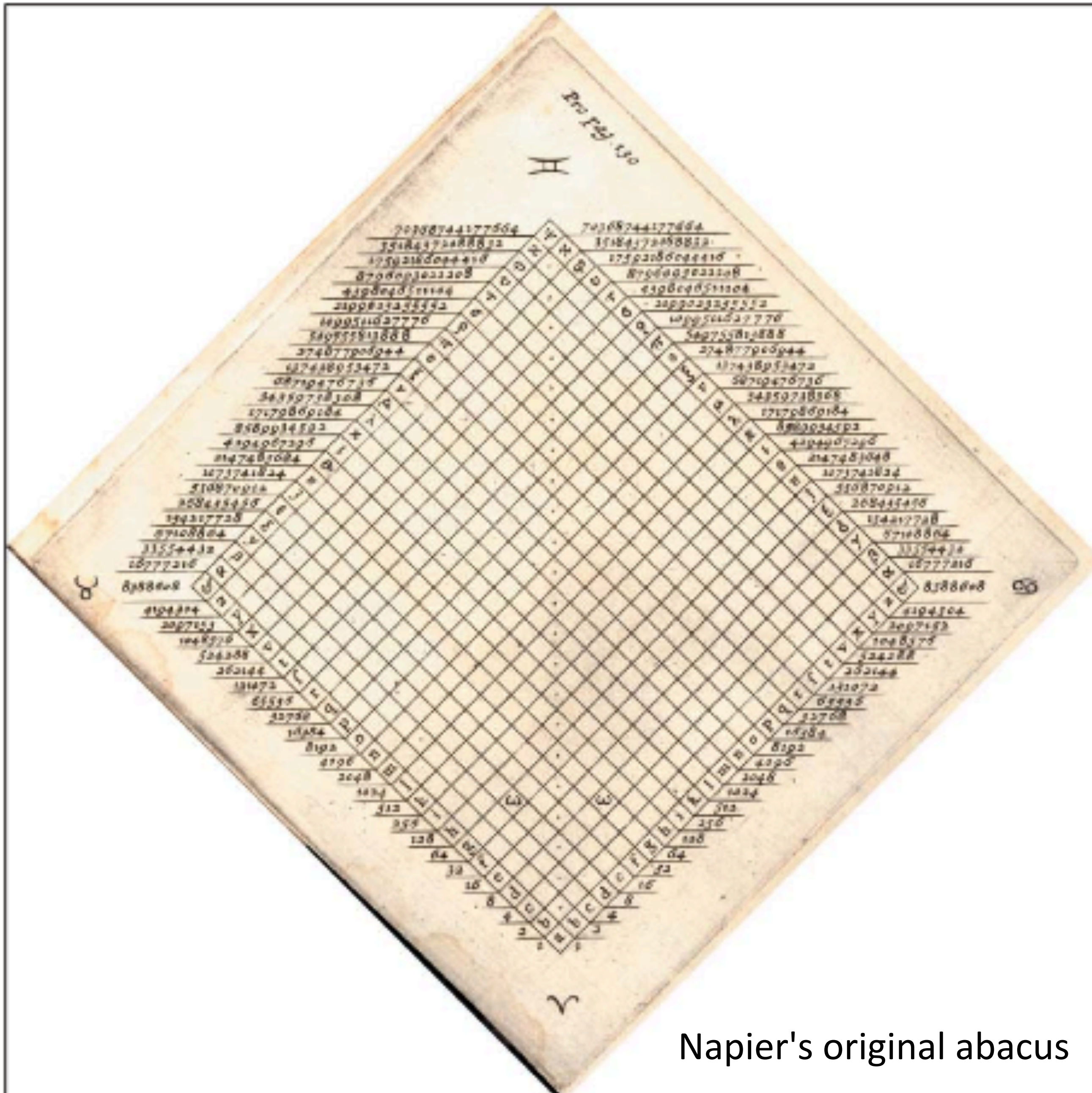

RABDOLOGIÆ,
SEV NVMERATIONIS
PER VIRGULAS
LIBRI DVO:
Cum Appendice de expeditis-
simo Multiplicationis
Promptvario.
Quibus accessit & Arithmeticæ
Localis Liber vnus.
Authore & Inventore Ioanne
Nepero, Barone Mer-
chistonis, &c.
Scoto.

EDINBVRGI,
Excudebat Andreas Hart, 1617.

Our belief can be justified by John Napier's construction of a "base two" checker-board abacus in his *Rabdologi*æ (1617).

In *Arithmeticæ Localis*, (the third part of his *Rabdologiæ*), the way that he shows how to record numbers in base two could easily be changed into computation in other bases (Gardner, 1986).

And because computation performed on such abaci do not require number words, the same methods of computation could be accompanied by very different systems of numerals, reflecting, for example, different systems of units of measurement.



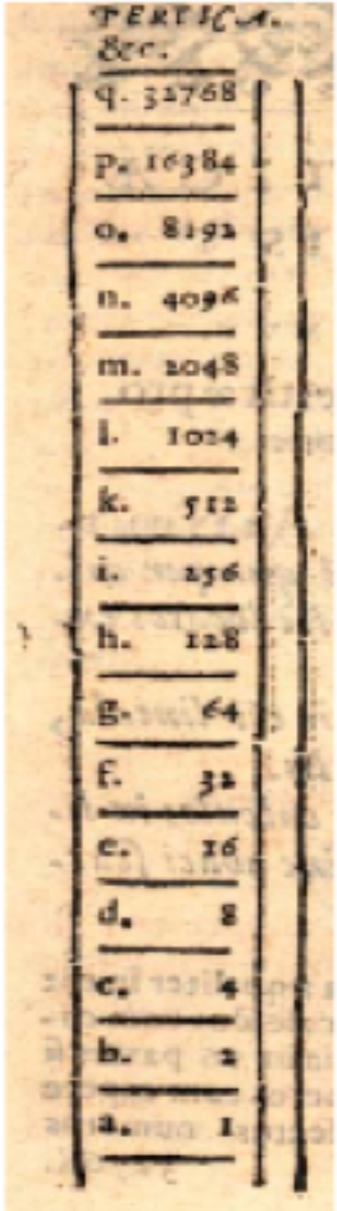
Napier's original abacus


2048
1024
512
256
128
64
32
16
8
4
2
1
2048
1024
512
256
128
64
32
16
8
4
2
1

Napier's original abacus

His abacus was a square checkered board that can be extended up and to the left, where each square has a local value as shown below:

2^n*2^m							2^m
			64	32	16	8	4
			32	16	8	4	2
2^n			16	8	4	2	1


TERTICA.
&c.

q.	32768
p.	16384
o.	8192
n.	4096
m.	2048
l.	1024
k.	512
i.	256
h.	128
g.	64
f.	32
e.	16
d.	8
c.	4
b.	2
a.	1

Napier did not have the concept of "base 2". He used a different notation: a = 1, b = 2, c = 4, d = 8, ...
So, for example, 37, which in base 2 is

$$100101 = 2^5 + 2^2 + 1,$$

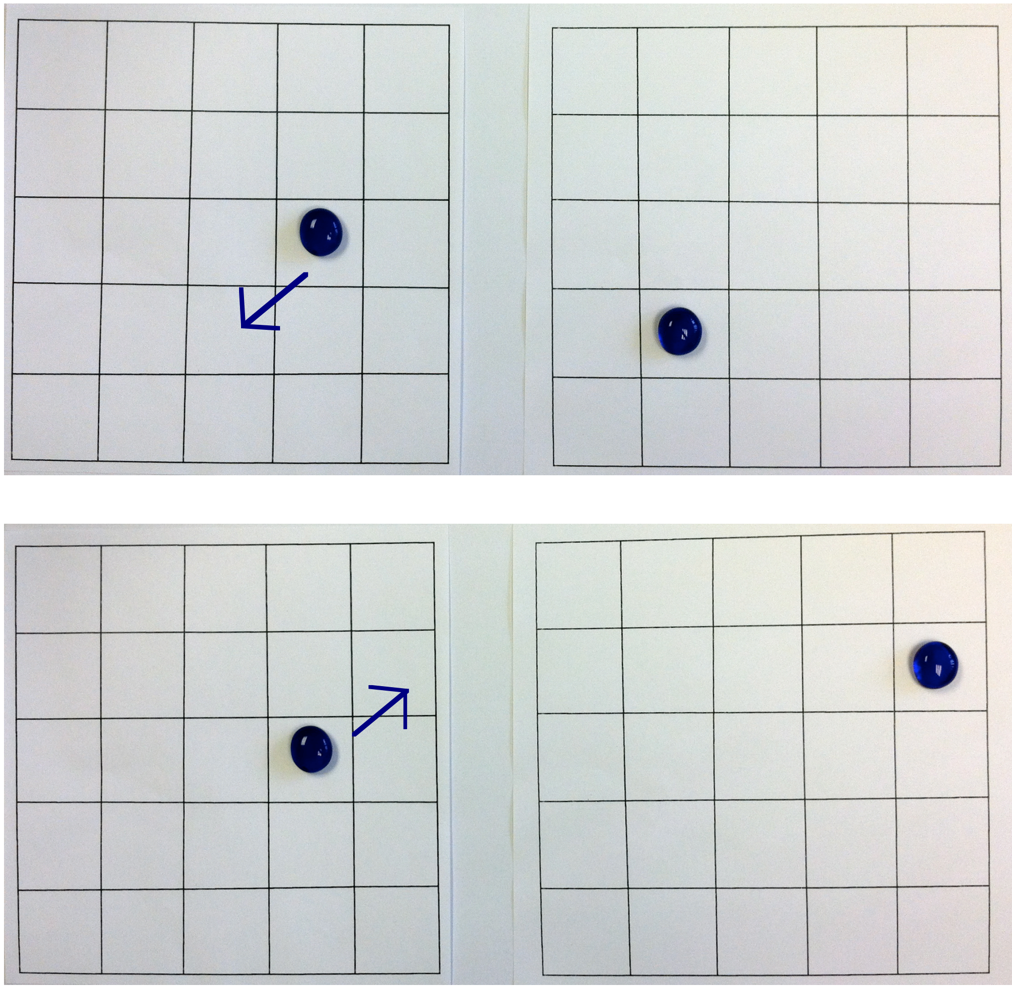
was written as

f c a.

Napier showed how to change from decimal notation to his notation and back, and how to compute the sums, differences, products, quotients and square roots on his board.

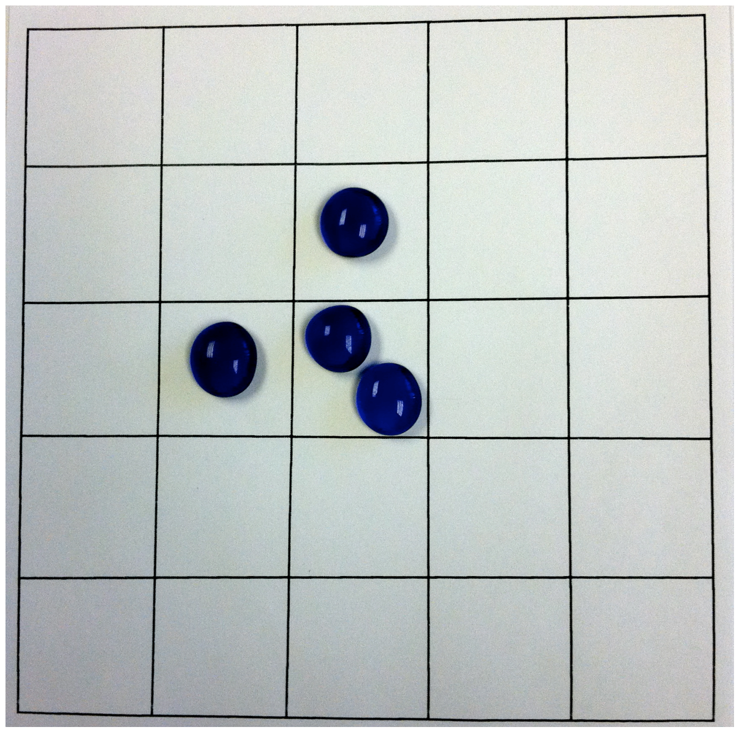
The rules were simple. On each square you may put one token

or two tokens.



A token can always be moved along a "short" diagonal. (The value remains the same.)

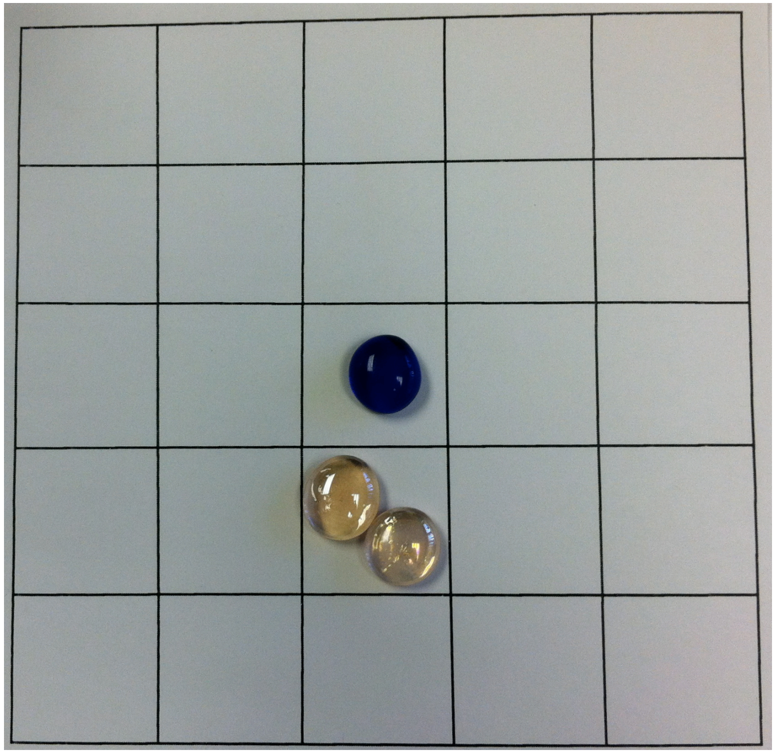
When you move a token to the right or down, it doubles.
A move in the opposite direction requires that two tokens be replaced by one.

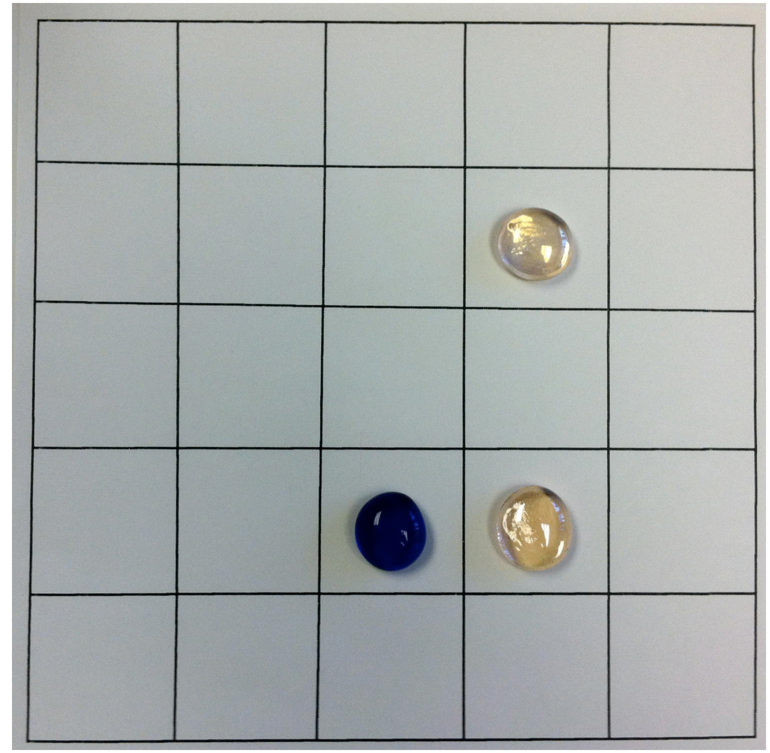


Thus the values in each square above are all equal.

As mentioned small modifications in the rules of moving tokens facilitate computations in any other, not too big, base.
And this shows that experimenting with different bases could have been a rather easy task, after the basic idea of "processing records" was established.

Let’s look at two rules for moving tokens.





Vertical rule: the value of the blue token and the values of the two yellow tokens together are the same.

Horizontal rule: the value of the blue token and the values of the two yellow tokens together are the same.

These rules correspond to the following local values on a board:

1000	200	40	8
500	100	20	4
250	50	10	2
125	25	5	1

And they are sufficient to carry out computation in any base that is the product of a power of 5 and a power of 2 (i.e. 5, 10, 20, 25, ...).

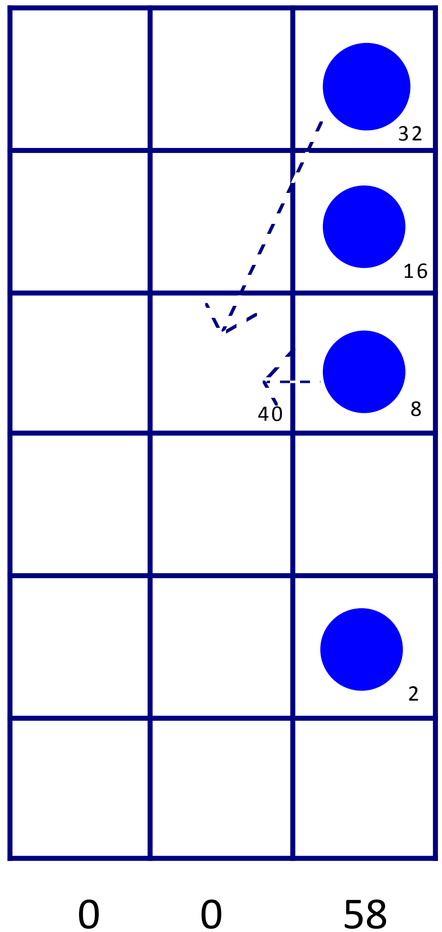
Examples.

First we show how to enter a number in base two into the right-most column and convert it into base 10.

We convert a number in base 2 into decimal using a decimal abacus.
The number in the last column is 111010 = 32 + 16 + 8 + 2 = 58
Consecutive configurations are shown together with the sum of the values in three columns.

800	160	32
400	80	16
200	40	8
100	20	4
50	10	2
25	5	1

800	160	32
400	80	16
200	40	8
100	20	4
50	10	2
25	5	1



800	160	32
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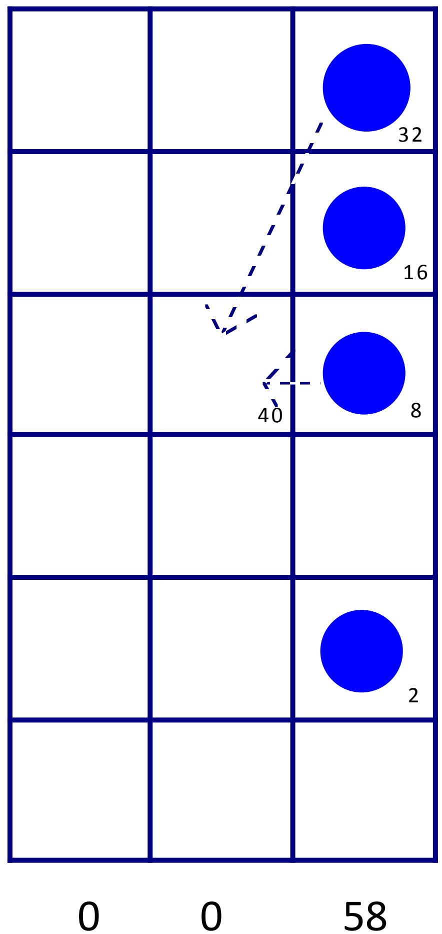

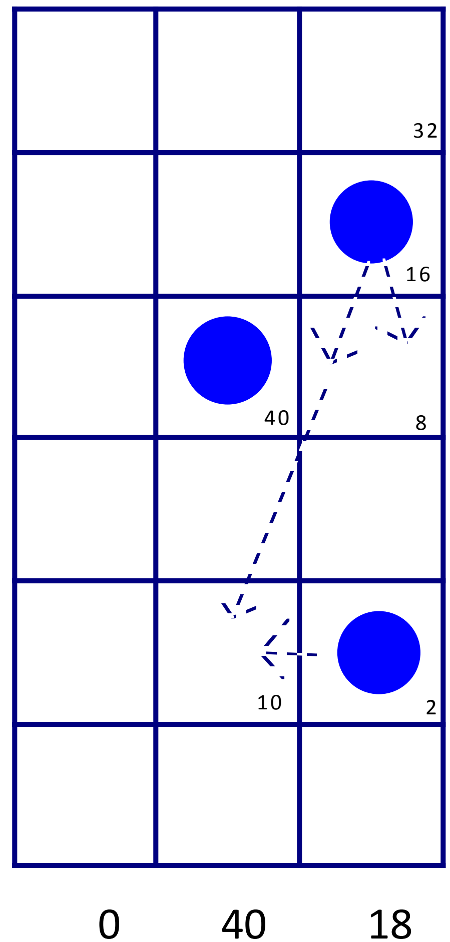

800	160	32
400	80	16
200	40	8
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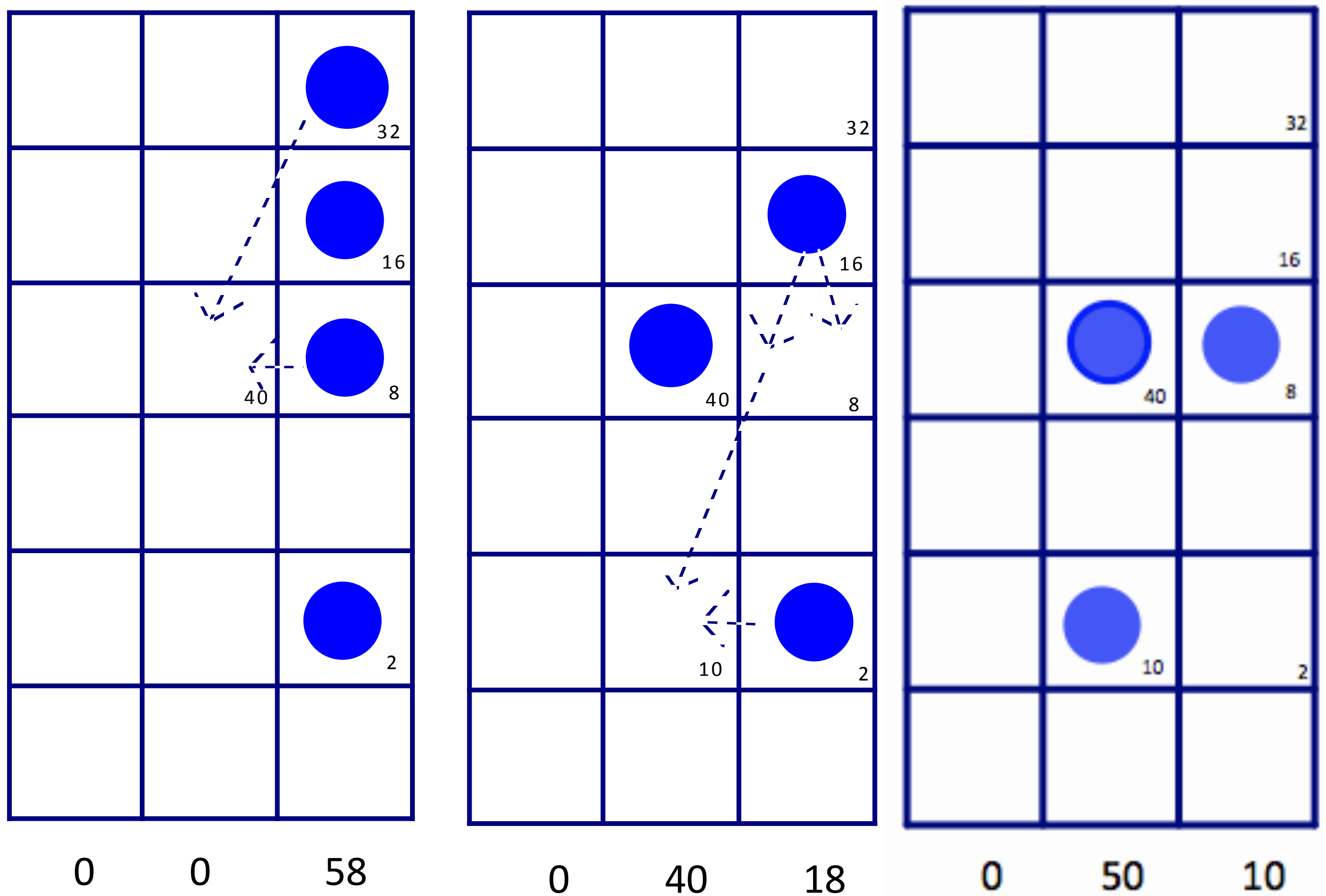
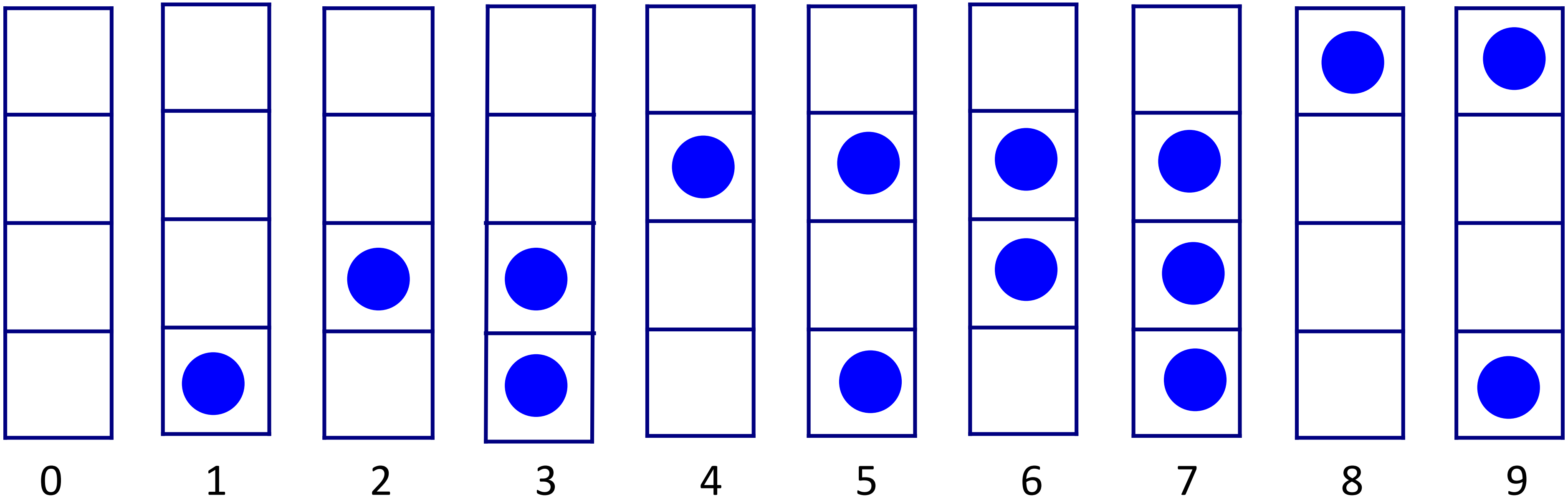
50	10	2
25	5	1

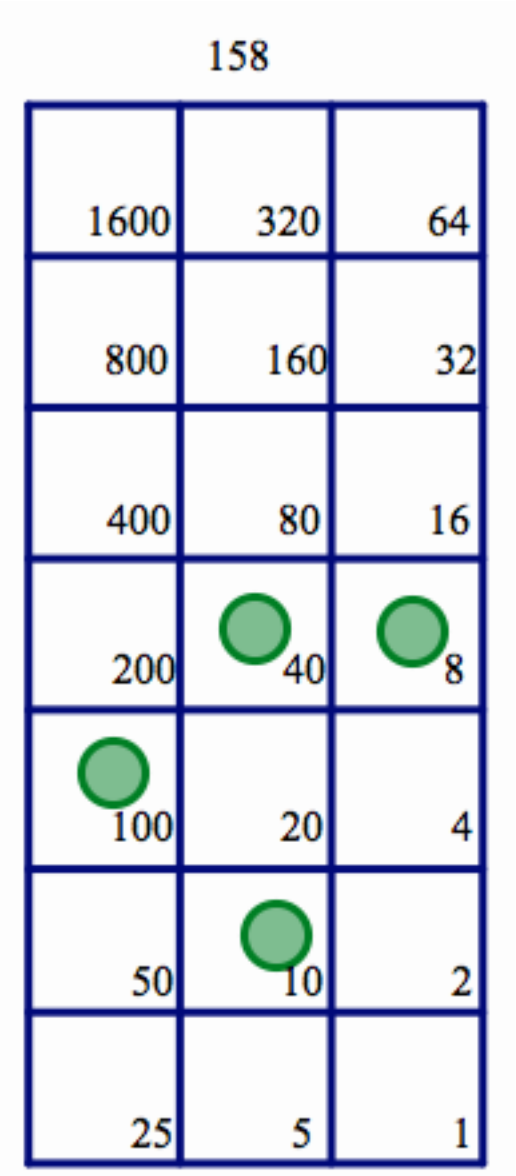


This can be read as 58.

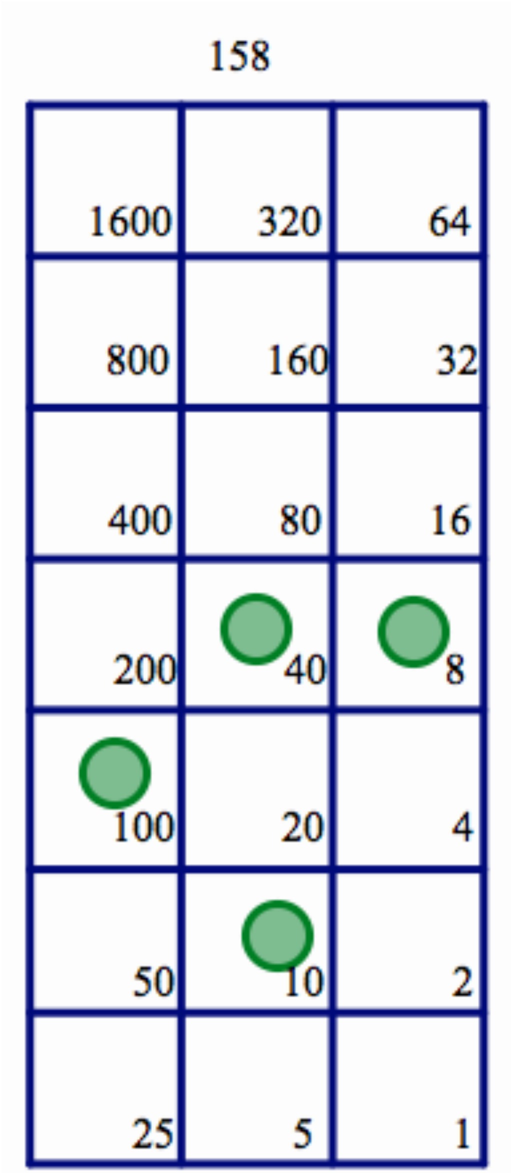
How to represent the digits 1, ... , 9 in one column (the lowest location is the power of ten):

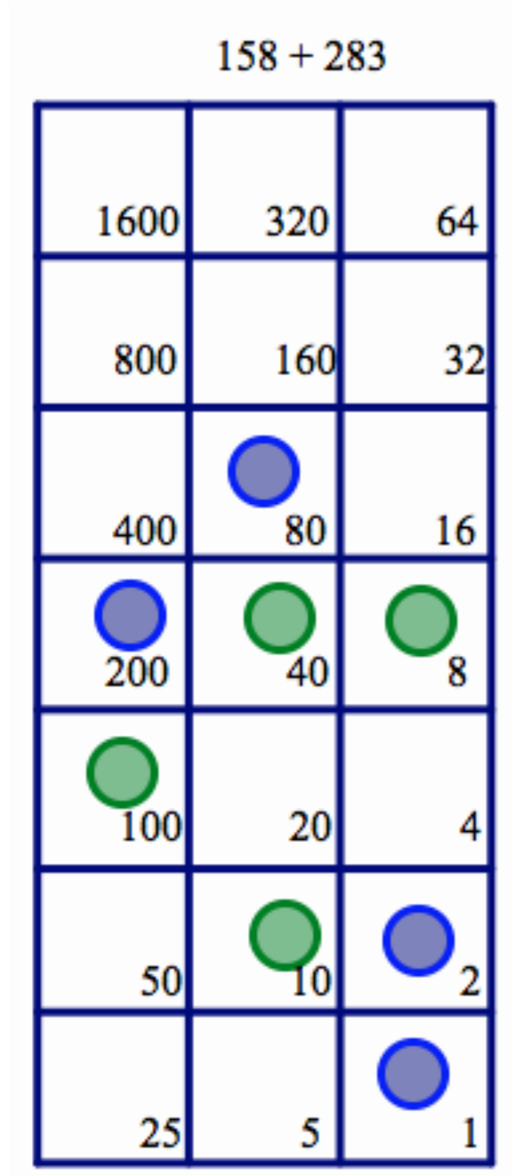


Example of how to add two numbers in base 10: 158 + 283

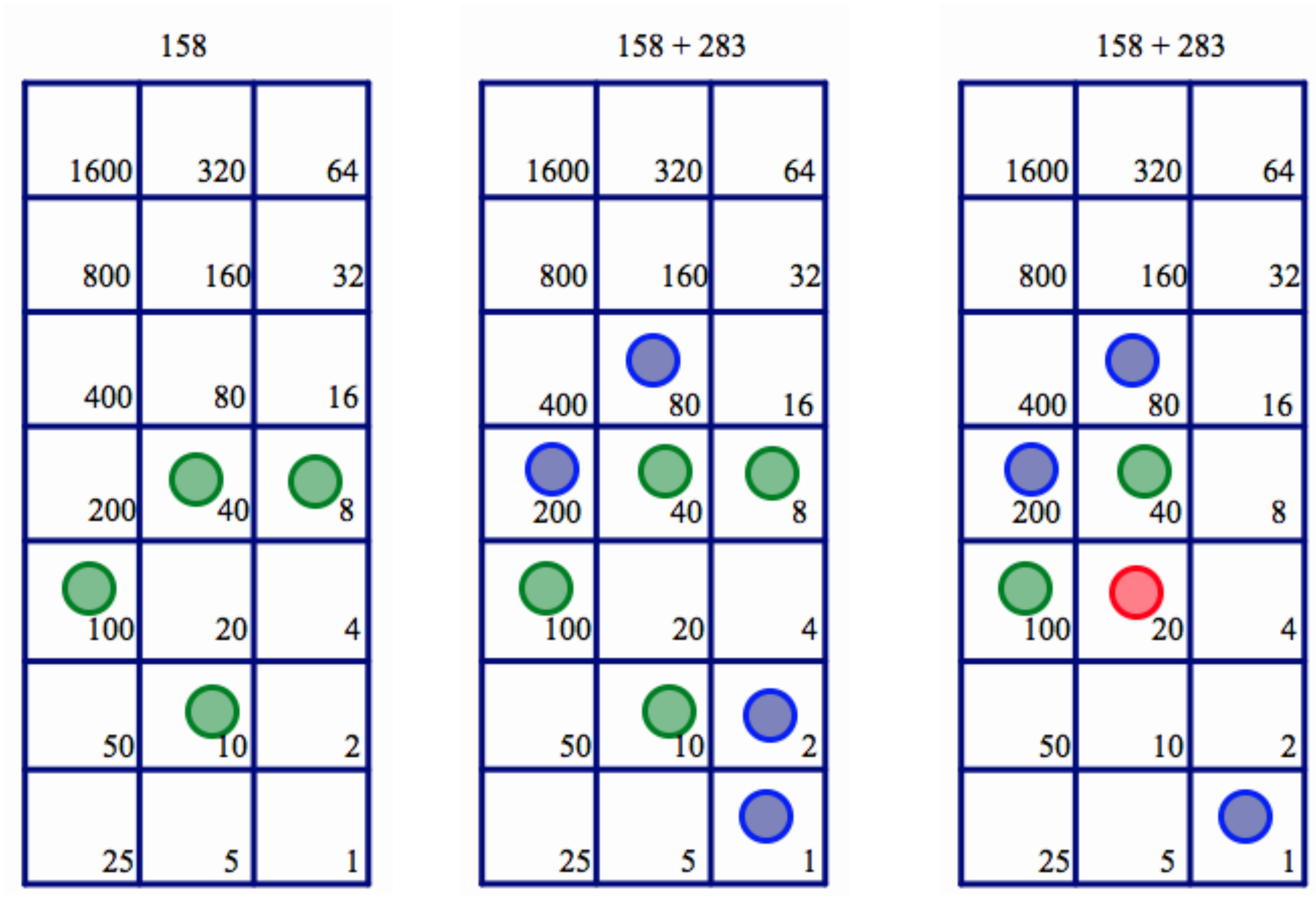


Example of how to add two numbers in base 10: 158 + 283

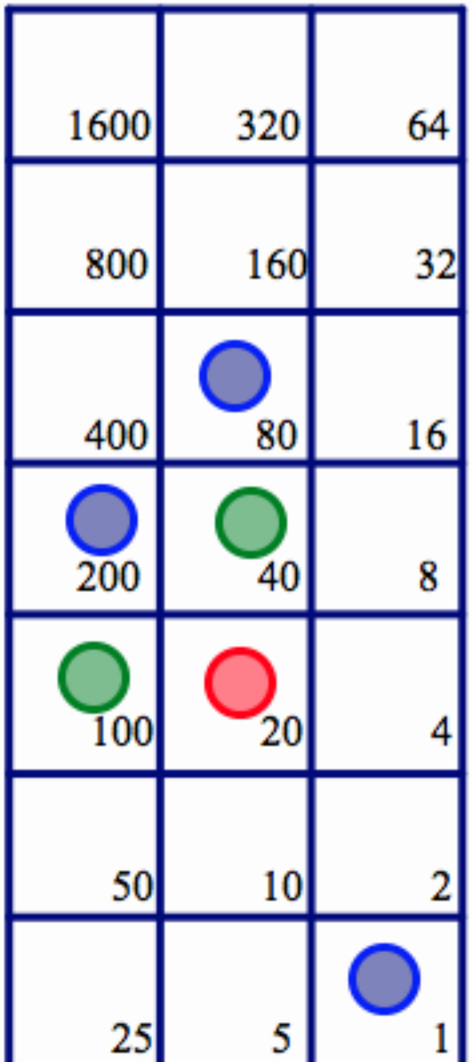




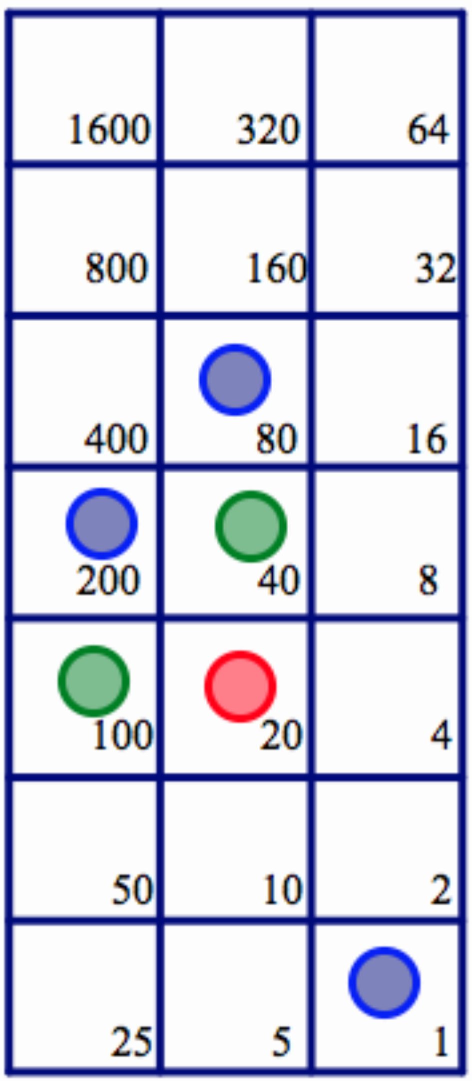
Example of how to add two numbers in base 10: 158 + 283



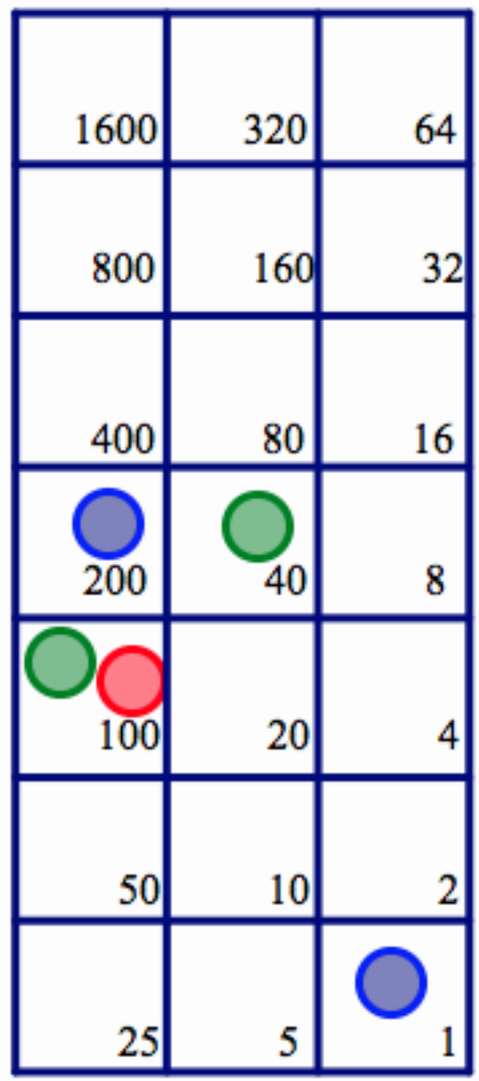
158 + 283



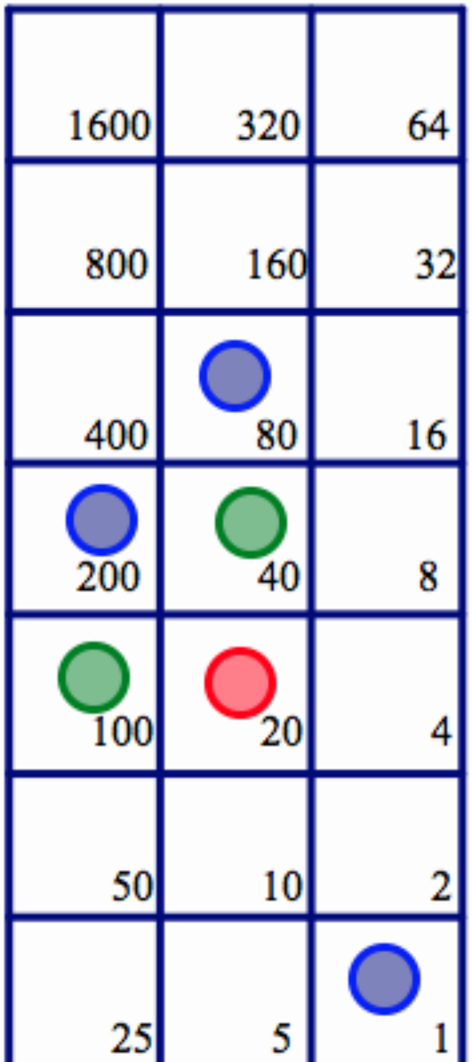
158 + 283



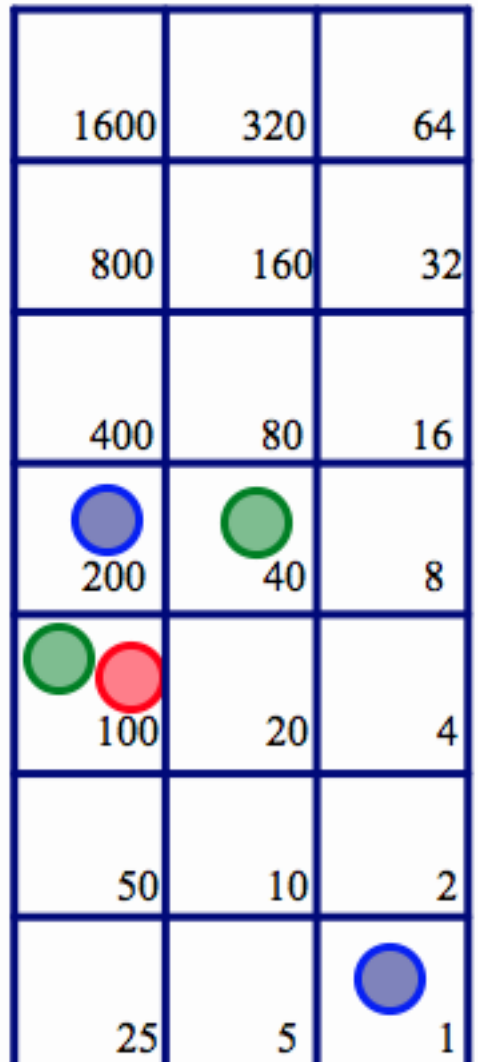
158 + 283



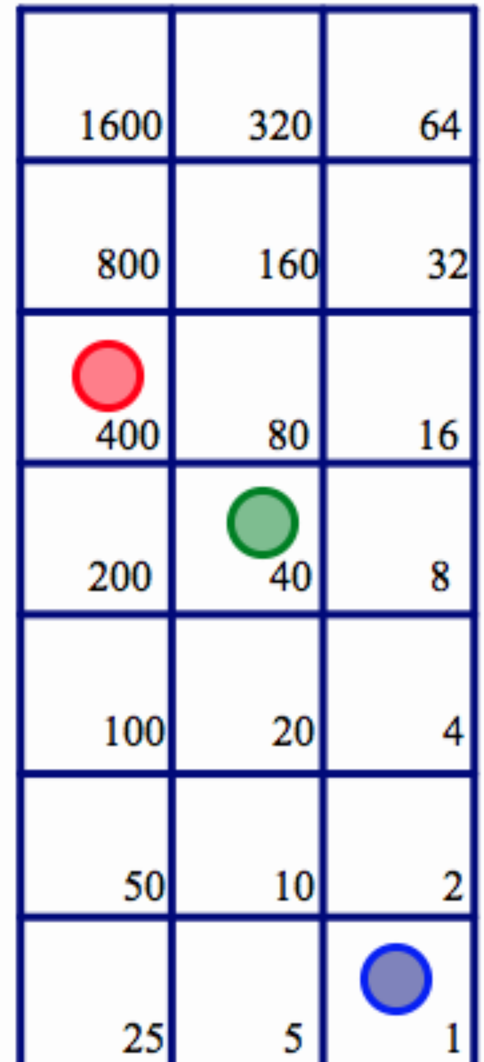
158 + 283



158 + 283



158 + 283 = 441



Similar rules can be shown for bases 6, 12. ..., and the "hybrid" base 60, which has both 3 and 5 as a factors.

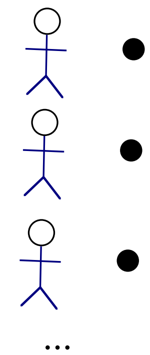
Two conclusions.

Our first conclusion is that it is possible (by the work of Napier) that developing the rules of arithmetic and developing number words were two very different processes that could have gone on in parallel without influencing each other very much. Or even that the rules of arithmetic *preceded* number words, and were formulated in terms of changing configurations of tokens.

Our second conclusion (as suggested by Leslie) is that arithmetic may be based not only on the process of counting, but also on the process of doubling (and its inverse, halving) which underlies the recording of numerosities.

Summary of our hypotheses

Here is a possible logical sequence of events:



1. Representation of collections by a one-to-one matching with tokens.

2. Discovery that sets of tokens are either even or odd, but not both (Leslie).

3. Creation of “binary” records of numerosities (Leslie). o | | o | ///
(These records are the first numbers.)

4. Processing the records as patterns of tokens on counting boards (Napier) (Different bases are experimented with.)

5. Number words describing abstract numbers are created, and they spread through societies.

3. The current approach to early learning of arithmetic

In grades K-3, the introduction to arithmetic starts with children memorizing a sequence of number words up to 20 or more.
They also learn to count small collections of "things" by matching them with the memorized sequence of words.
"The last word tells how many.”
Next, addition is introduced as joining two collections of objects and counting the total, which is improved to counting up from the bigger number, and skip counting.
These activities are preliminary to learning more "standard" methods in the future.
A similar pattern is repeated when multiplication is introduced as repeated addition.

A theoretical justification for this approach seems to be very recent, i.e., the work of Dedekind (1883; 1963) and Peano (1889; 1967) in the late nineteenth century.
They expected that the recursive definition of addition in terms of "next number" and of multiplication in terms of addition would allow one to deduce all the properties of addition and multiplication from the simple properties of "next number".

We know now that properties of addition and multiplication cannot be derived from the properties of "next".
But this fact belongs to the history of number theory and logic, and not to school math, because logical deductions among different properties of numbers are not part of elementary mathematics.

But basing addition on counting and multiplication on repeated addition has important consequences for teaching in early grades.

The reason for this is that this approach leads to very inefficient methods of computation.

For example, computing the product of two three-digit numbers by repeated addition requires several hundred steps, while other methods require no more than 20 steps.

4. An alternative approach to introducing numbers

If we would follow the idea of Leslie, we would start, not with counting as the only operation, but with two operations, counting and doubling.

This would allow children to create records of numbers (much bigger than 20), which in turn can be processed in a variety of ways on very simple counting devices.

At present in early grades children see in their classrooms a "number line",

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

When we use two basic operations, "next" and "double", the number line is replaced by a "number heap":

1															
2								3							
4				5				6				7			
8		9		10		11		12		13		14		15	
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31

which, when read horizontally, gives the next number, and when read vertically gives the double.

This approach allows one to introduce large numbers much earlier.
For example:
The question, what is 40?, can be answered as follows:

5 is the double of 2, plus one,
10 is 5 doubled,
20 is 10 doubled
and 40 is twenty doubled

If you look at the heap, we see that any number up to 31 can be obtained from 1 by no more than four applications of "double" and "double plus one".

1
2 3
4 5 6 7
8 9 10 11 12 13 14 15
16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31

A small abacus board

200	40	8
100	20	4
50	10	2
25	5	1

is sufficient to do addition and subtraction up to 200, and also to find all products up to 200 of two smaller numbers.
And neither task requires that children memorize addition, subtraction, or multiplication facts.

Final conclusion

We don't suggest that the approach presented should be used in schools, but only that looking for alternatives to the existing method is needed.
And that the history of arithmetic may provide valid guiding principles.

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