The "Ladder and Box" Problem: From Curves to Calculators

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Outline

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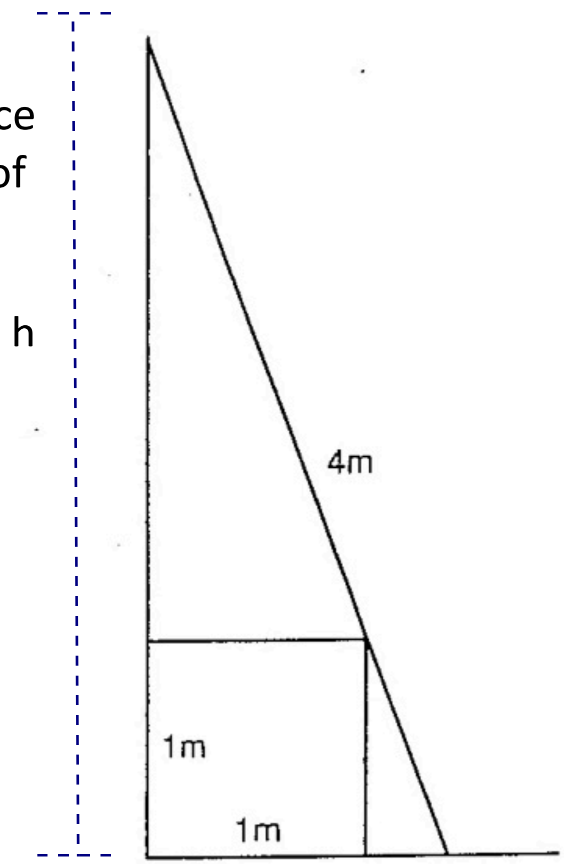
2. A modern solution using a simple trigonometric equation
3. How the problem has been solved historically, and what it meant to “solve” a problem
4. Final remarks

1. Introduction: A “new” problem with historical roots, and its mathematical formulation

399. The Ladder and the Box A ladder, 4 metres long, is leaning against a wall in such a way that it just touches a box, 1 metre by 1 metre, as in the figure. How high is the top of the ladder above the floor?

What distance h is the top of the ladder above the floor?

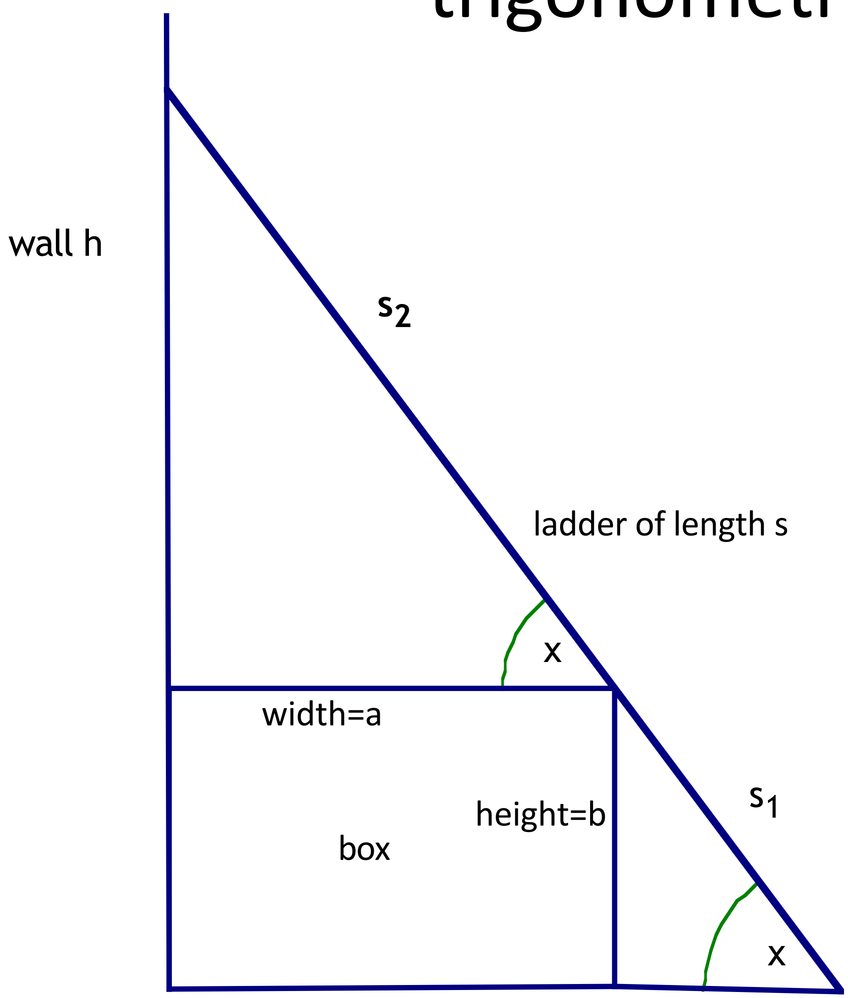


The “ladder and box” problem (this example from Wells, 1992) is relatively new; it first appeared in A. Cyril Pearson’s 1907 *20th Century Standard Puzzle Book* (London).

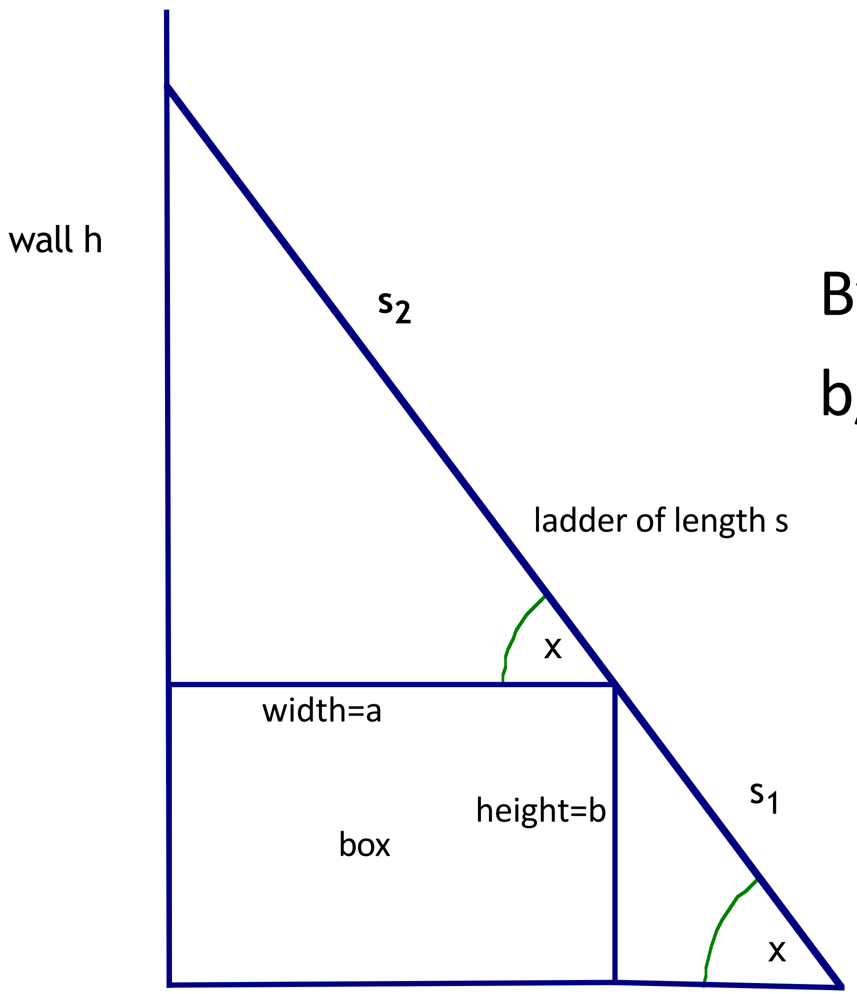
But its mathematical underpinnings have been traced back to Nicomedes (~200 BCE), as well as to Newton (1720) and Thomas Simpson (1745).

The problem is to build a right triangle, given a right angle, and an inscribed rectangle (here, the rectangle is a square), together with a segment s, representing the length of a hypotenuse.
In puzzle problems, one is asked to find the length of the leg of the triangle.

2. A modern solution using a simple trigonometric equation



At present, the lengths of the legs of the right triangle can be found analytically by solving a simple trigonometric equation that is easy to derive.



So we have

$s_1 * \sin(x) = b$

$s_2 * \cos(x) = a$

$s_1 + s_2 = s$

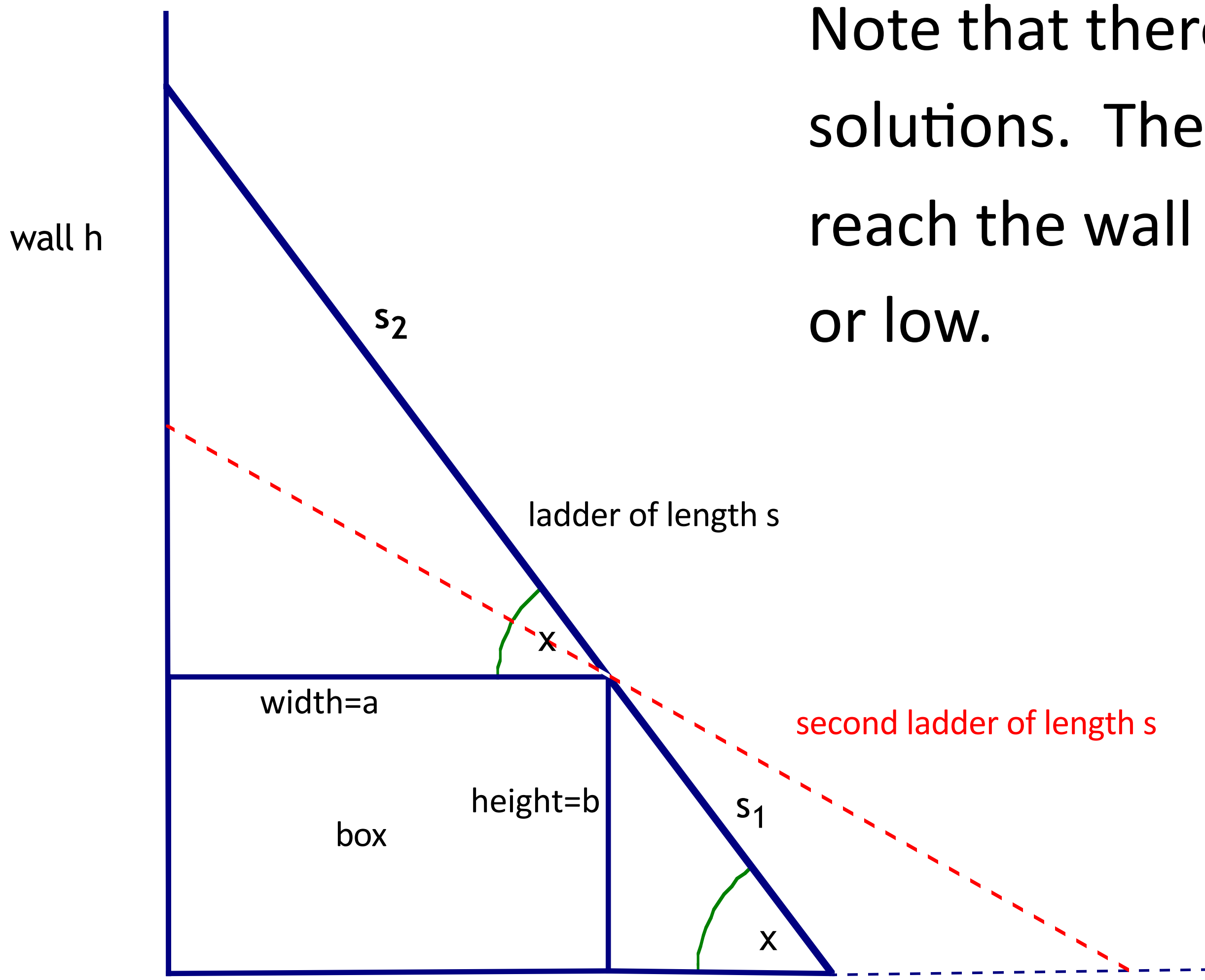
By eliminating s_1 and s_2, we get

$b/\sin(x) + a/\cos(x) = s$

We know a, b, and s.

Both solutions to this equation can be easily found, for example, by using SOLVER on a graphing calculator. Then the required height h is

$h = s * \sin(x)$.



Note that there are two solutions. The ladder can reach the wall either high or low.

When b = a (the box is a square), the trigonometric equation b/sin(x) + a/cos(x) = s can be reduced to two quadratic equations that can be solved using only square roots:

Let $y_1 = \sin(x)$ and $y_2 = \cos(x)$. So we have $\frac{a}{y_1} + \frac{a}{y_2} = s$.

Therefore we have: ${y_1}^2 + {y_2}^2 = 1$

$$y_1 * y_2 = \frac{a}{s}(y_1 + y_2).$$

Because $(y_1 + y_2)^2 = {y_1}^2 + {y_2}^2 + 2(y_1 + y_2)$, we have

$$(y_1 + y_2)^2 = \frac{2 * a}{s} * (y_1 + y_2) + 1.$$

Let $z = y_1 + y_2$. We can find two values, z_1 and z_2 of z by solving for z this quadratic equation:

$$z^2 = 2 * \frac{a}{s} * z + 1$$

Now for each z_i we have

$$y_1 + y_2 = z_i$$

$$y_1 * y_2 = \frac{a}{s} * z_i.$$

So y_1 and y_2 can be found by solving for y this equation:

$$y^2 - z_i * y + \frac{a}{s} * z_i = 0.$$

There are similar, but different ways to solve this problem.

For example, see Thomas Simpson’s 1745 solution (below); and see also J.V. Uspensky’s 1948 solution in his book *Theory of Equations*, and B. Fisher’s 1972 solution in his Mathematics Magazine article, *The Solution of a Certain Quartic Equation*.

3. How the problem has been solved historically, and what it meant each time "to solve a problem"

a. Geometric solutions
b. Algebraic solutions
c. Solutions as puzzles or as recreational mathematics

This problem is interesting from the point of view that its general case, finding a solution for an arbitrary inscribed rectangle with sides a and b, was considered to be very difficult.
(It cannot be constructed with straight edge and compass, and the polynomial equation expressing the lengths of the legs in terms of a, b, and s, has degree four.)

The main simpler version of the problem is when the rectangle is a square, which admits other techniques that don't work in the general case.

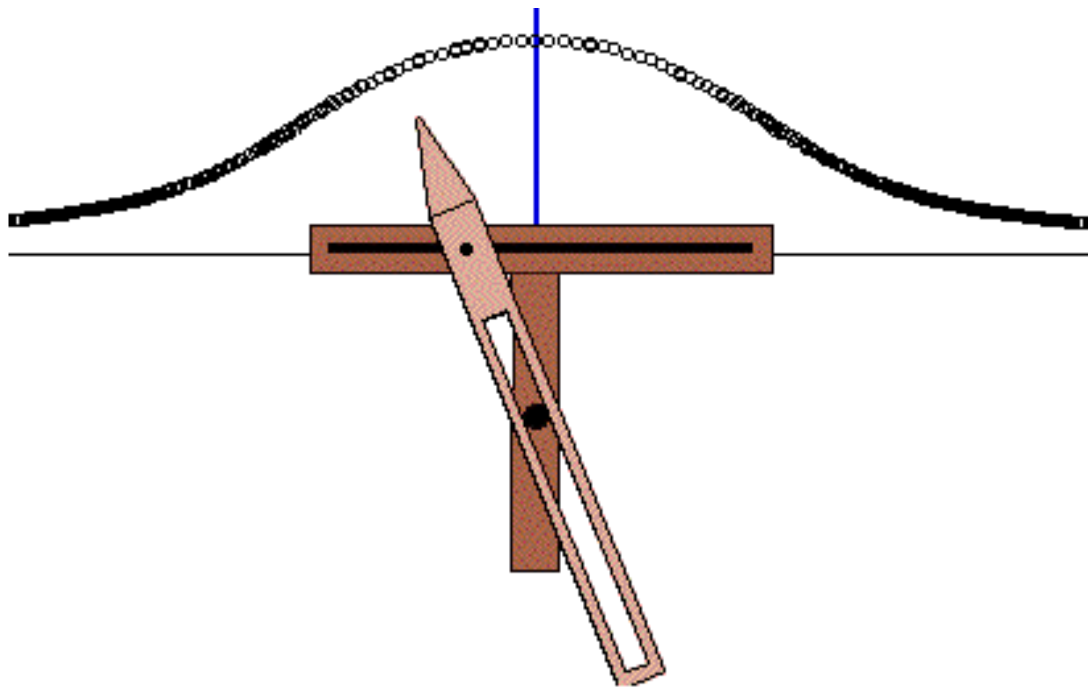
The other special cases of the problem in which one can rather easily find the lengths of the legs of the required right triangle are when all the numbers involved, namely, the sides of the rectangle, the triangle's hypotenuse, and the two legs to be found, are rational. They often pop up in puzzle books and recreational mathematics, because they can be solved by "guess and check" methods.

a. Geometric solutions (at the time of the Greeks).

We credit this information to Audun Holme*, Geometry: Our cultural heritage* (2010).

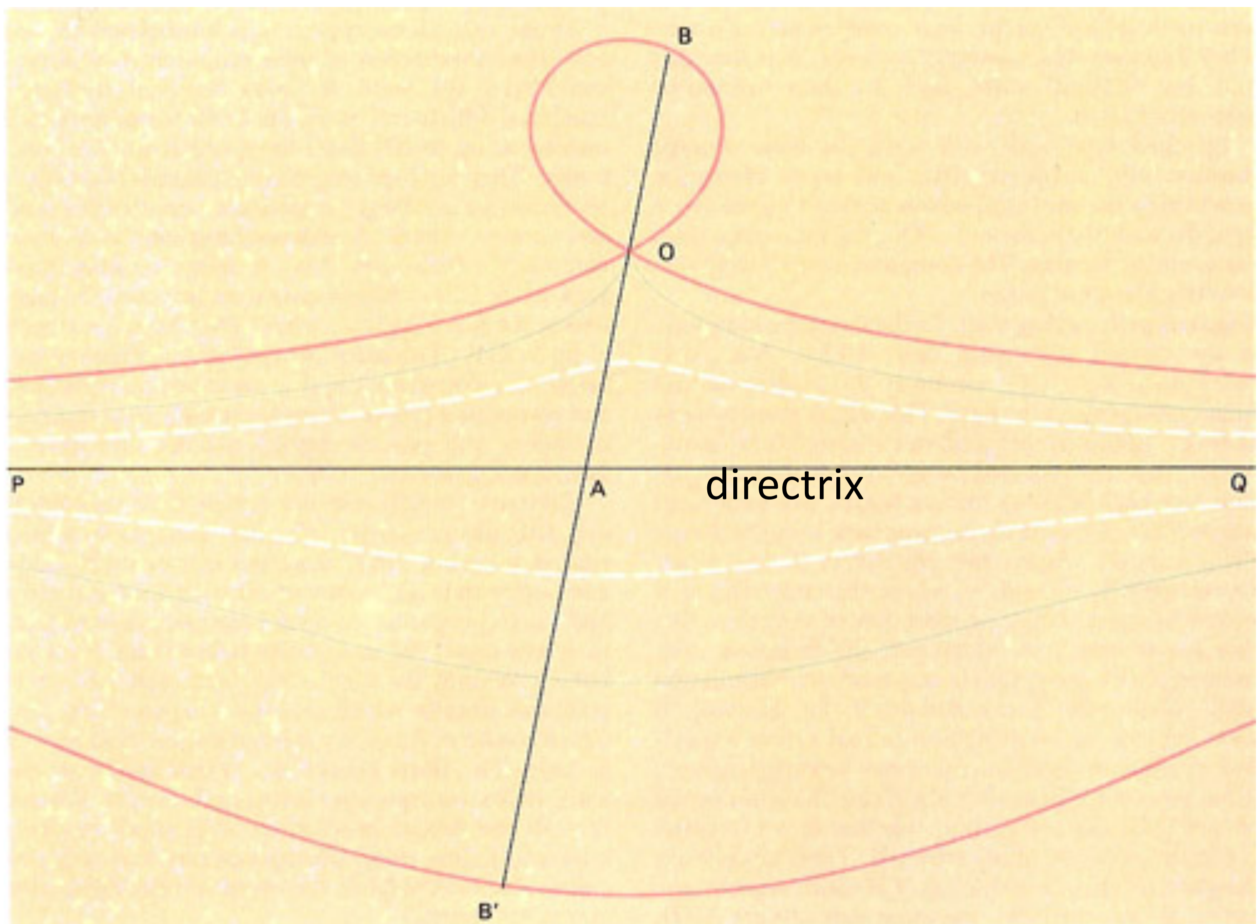
For an arbitrary rectangle, the problem cannot be solved by a straight edge and compass construction. But it can be solved by using Nicomedes' (~280-210 BCE) conchoid and tools related to it.

Here is a diagram of a tool that can be used to draw a conchoid:

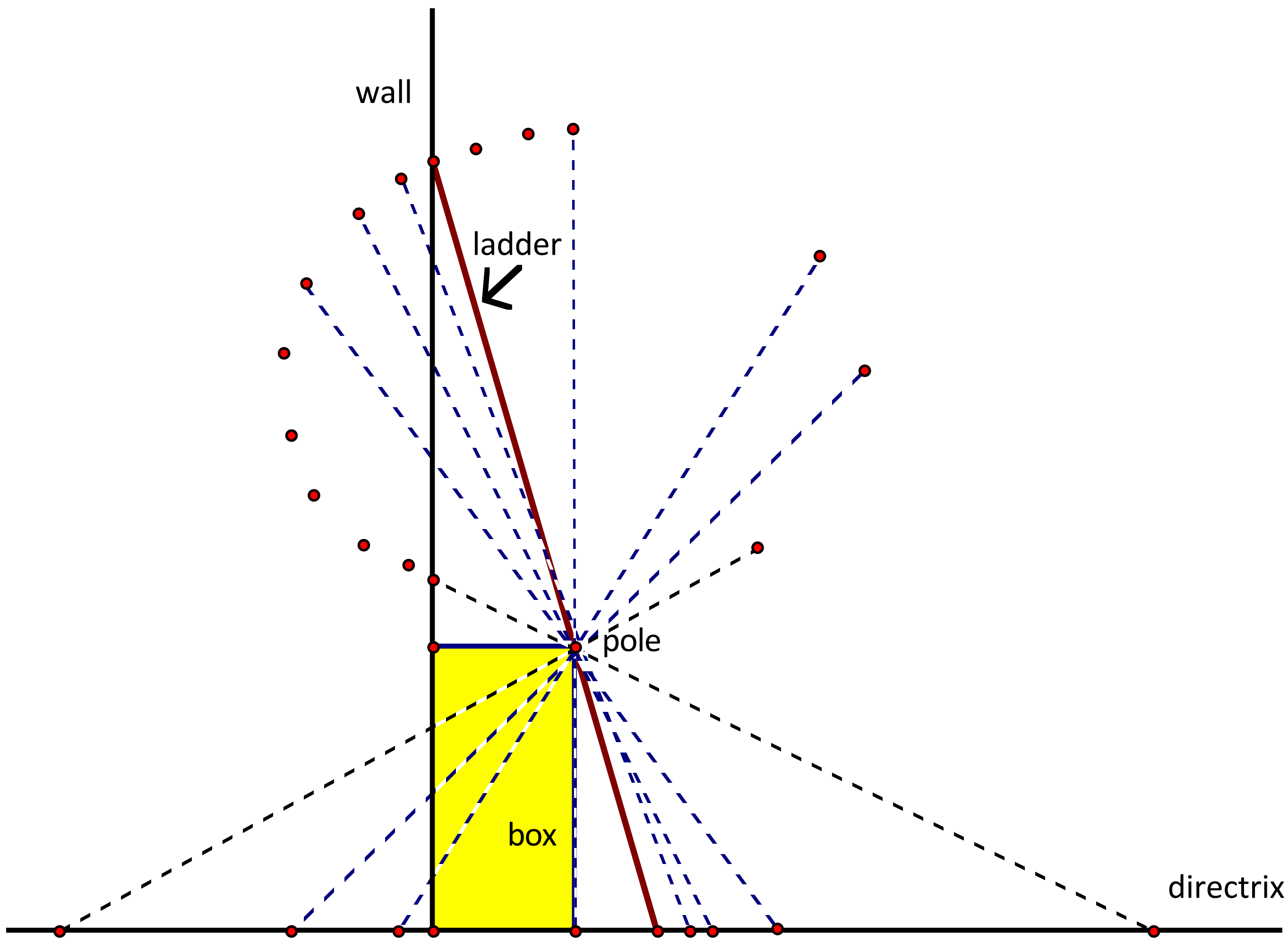


This tool will draw both parts of the conchoid.

Here we show both parts with the “loop” above rather than below the directrix (the horizontal line PQ):



But because for our problem we need only ONE point on a curve, a simple straight edge with two marks suffices:



At that time, the solution to a problem was a step-by-step description of a geometric construction using only specified “tools”. But these “tools” seem to represent rather abstract operations.

For example, using a “straight edge” meant that you can construct a unique straight line passing through any two given points, and using a “marked” ruler meant that you can put a point on a conchoid, given a point on its directrix and its pole.

No numbers were involved in any construction.
The length of a segment was not a number, but the segment itself.

b. Algebraic solutions

Both Isaac Newton in *Universal Arithmetic* (1720) and Thomas Simpson in *A Treatise of Algebra* (1745) analyzed a large number of geometric problems, showing how they can be solved using algebraic techniques.

These solutions did not involve analytic geometry, because no coordinate systems were involved.

Instead, the relationships among segments, such as proportions, were translated into equations involving lengths of those segments.

Then, step-by-step procedures for solving these equations were shown.

Both authors considered only the special case (where the rectangle is a square, a=b), because it leads to a biquadrate equation.*

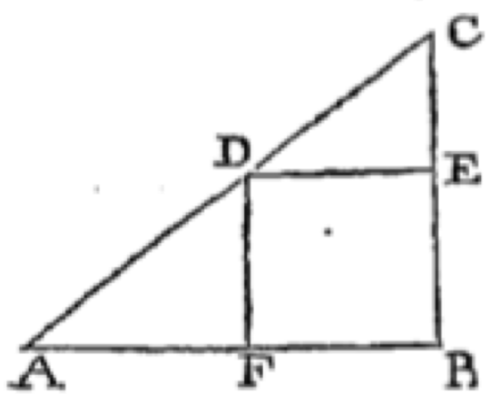
*A biquadrate, or biquadratic, equation is a quartic equation that can be solved with only square roots (no cubic roots are needed). (We showed such a solution in 2. above.)

Here is the problem in Thomas Simpson's 1745 book:

PROBLEM XV.

The Side of the inscribed Square BEDF, *and the Hypothenuse* AC *of a right-angled Triangle* ABC *being given; to determine the other two Sides of the Triangle* AB *and* BC.

Let DE or DF $=a$, AC $=b$, AB $=x$ and BC $=y$; then it will be as $x:y::x$ $-a$ (AF) : a (FD) whence we have $ax=yx-ya$, and consequently $xy=ax+ay$. Also we have $AB^2+BC^2=AC^2$, or $x^2+y^2=b^2$ (by *Eu.* 47.1), to which Equation let the double of the former be added, and there arises $x^2+2xy+y^2=b^2+2ax+2ay$, that is $\overline{x+y}|^2=b^2+2a\times\overline{x+y}$, and consequently $\overline{x+y}|^2-2a\times\overline{x+y}=b^2$; wherefore, by considering $x+y$ as one Quantity, and compleating the Square, we have $\overline{x+y}|^2-2a\times\overline{x+y}+a^2=b^2+a^2$; whence $\overline{x+y}-a=\sqrt{b^2+a^2}$, and $x+y=\sqrt{a^2+b^2}+a$; which put $=c$, and then, by substituting $c-x$ instead of its Equal (y) in the foregoing Equation, $xy=ax+ay$, there will arise $cx-x^2=ac$; whence x will be found $=\frac{1}{2}c+\sqrt{\frac{1}{4}cc-ac}$ and $y=\frac{1}{2}c-\sqrt{\frac{1}{4}cc-ac}$.



In both cases a solution was a step-by-step algebraic procedure for computing the required numbers.

But the solution showed only the method of finding the numbers.

No specific numbers were involved, either in the formulation of the problem or in the procedure.

The method of showing algebraic procedures by starting with a specific case with numerical coefficients, and only then showing a general procedure, which is common in modern algebra textbooks today, was never used by Newton or by Simpson.

In applied mathematics, solutions to problems often include specific numbers.

But they are almost never limited to numbers.
Rather the opposite is true.

In applied problems we usually want to know more about the solution than we do in purely theoretical problems.

In a lecture she gave in 1969, Mary Cartwright talked about the fact that even when one looks just for a number, that is not all one wants to know, especially if one is an applied mathematician.

She wrote about solutions to differential equations, "…(one) really wants to know something about the solutions in general. Is there a periodic solution? Is it stable? Will it remain stable if I change a certain parameter? Will the period be longer or shorter?..."

(Cartwright, cited in Ayoub, 2004)

C. Puzzles and recreational mathematics

Here is the title page of A. Cyril Pearson's *20th Century Puzzle Book,* London (1907).

THE TWENTIETH CENTURY

STANDARD PUZZLE BOOK

THREE PARTS IN ONE VOLUME

EDITED BY

A. CYRIL PEARSON, M.A.

AUTHOR OF

'100 Chess Problems,' 'Anagrams, Ancient and Modern,' etc.

PROFUSELY ILLUSTRATED

SECOND IMPRESSION

LONDON

GEORGE ROUTLEDGE & SONS, LTD.

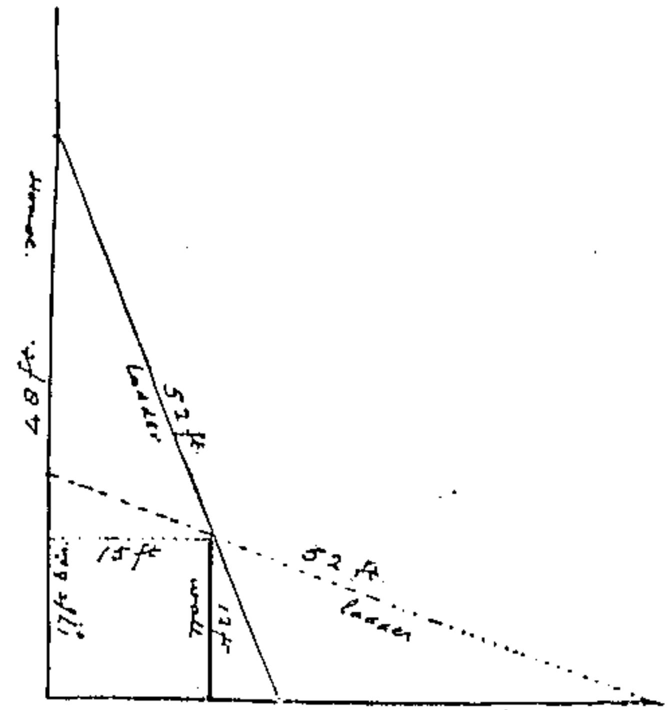
NEW YORK: E. P. DUTTON & CO.

And the first occurrence of the Ladder and box problem (that we could find):

The box is 15 ft wide and 12 ft tall, and the ladder is 52 ft long. It hits the wall in two places; the height of only one is given: 48 ft.

No. CIII.—CLEARING THE WALL

If a 52-feet ladder is set up so as just to clear a garden wall 12 feet high and 15 feet from the building, it will touch the house 48 feet from the ground.



Our diagram shows this, and also, by a dotted line, the only other possible position in which it could fulfil the conditions, if it were then of any practical use.

Recreational mathematics is for amateurs.

Problems are formulated, not in general, but in specific terms, and they are usually embedded in some kind of a story.

The solution to the ladder and box problem is just a number, independent of the method used to find it.

Also, in recreational problems, irrational numbers occur very rarely, and this explains why box and ladder problems have rational numbers as data and usually have rational solutions

All these examples fall into one category, which requires finding a rational root of a polynomial equation with rational coefficients.

And this problem can be solved by Euler's method (which really is a “guess and check” method with a bounded number of guesses).

But we did not find any author who discusses Euler's method in the context of this problem.

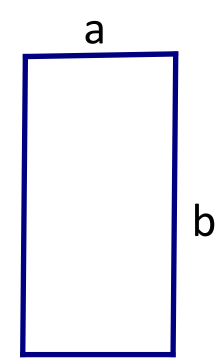
One thing is missing in all the examples we have seen:

How to design the puzzles so that the lengths needed are whole numbers.

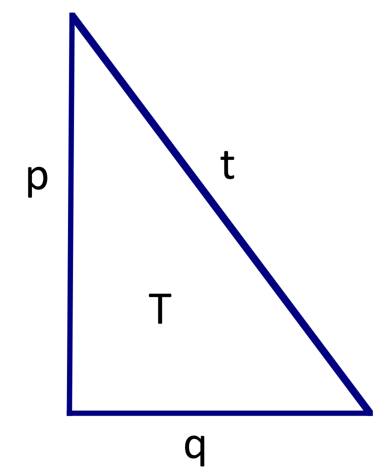
Puzzle makers don’t want to tell how to do it!

But if you’d like to know how, here’s a method.

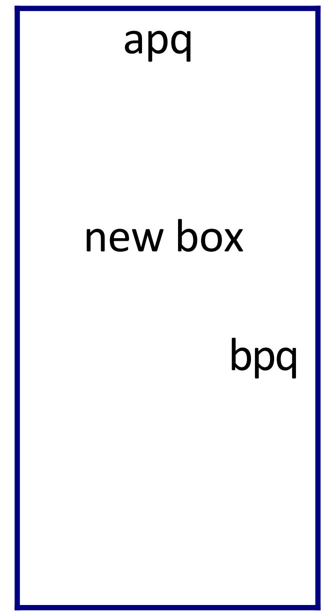
Start with a box whose dimensions are whole numbers a and b:



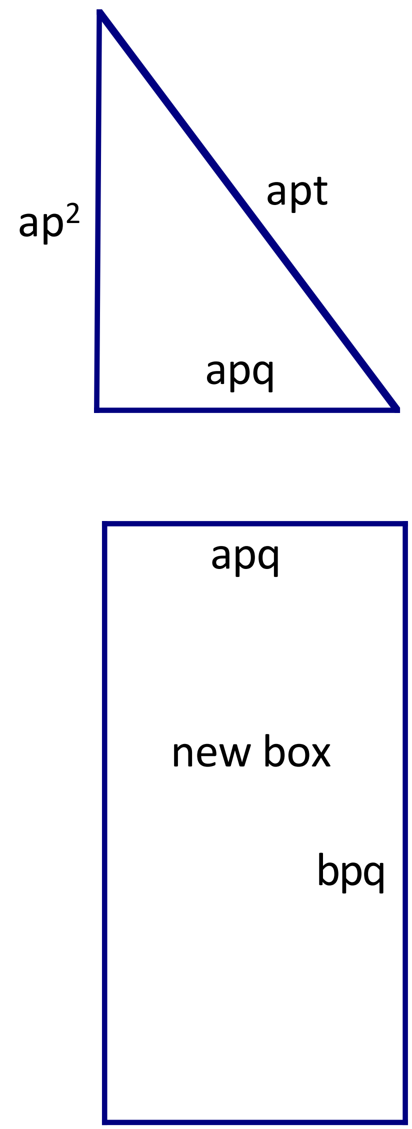
Now choose a Pythagorean triple, p, q, and t. Form the right triangle T with hypotenuse t and legs p and q:



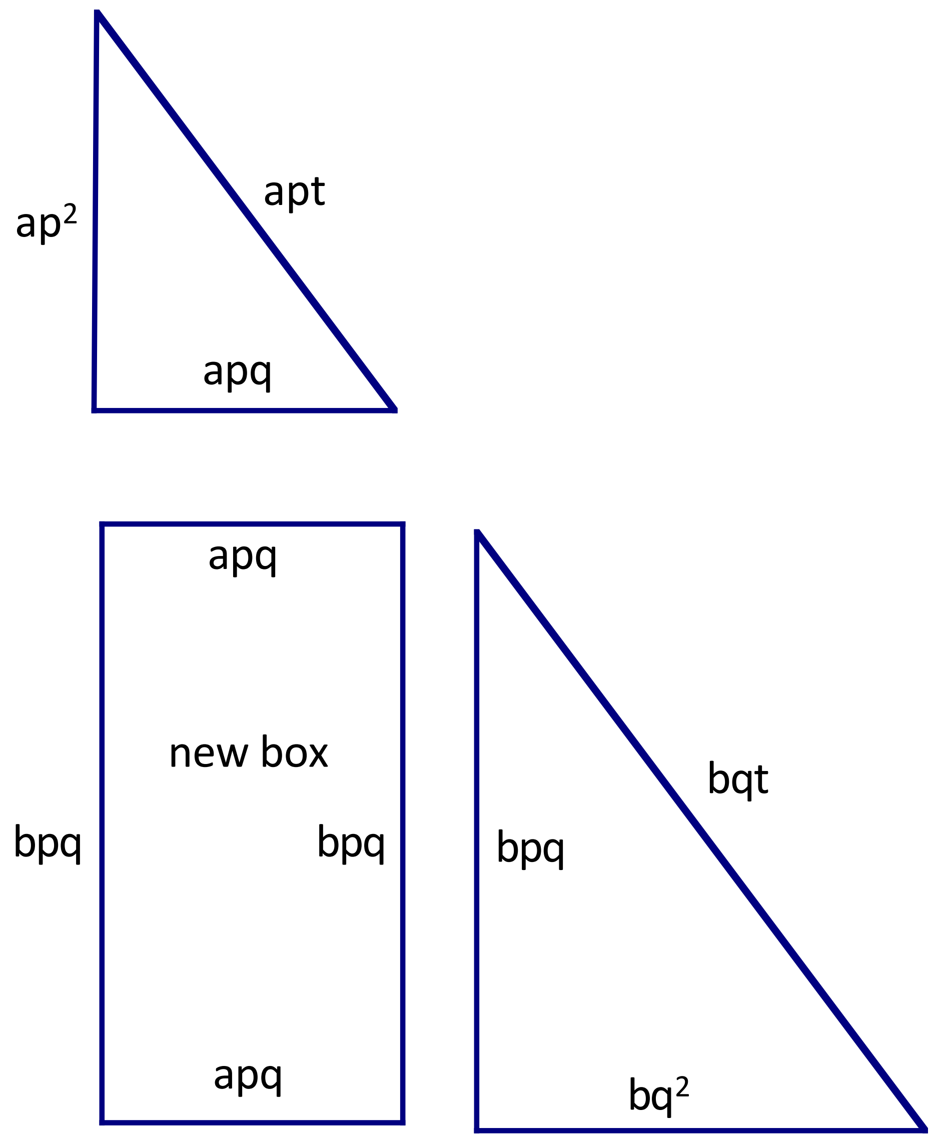
Form a new box with the shape
apq by bpq (similar to the first box).



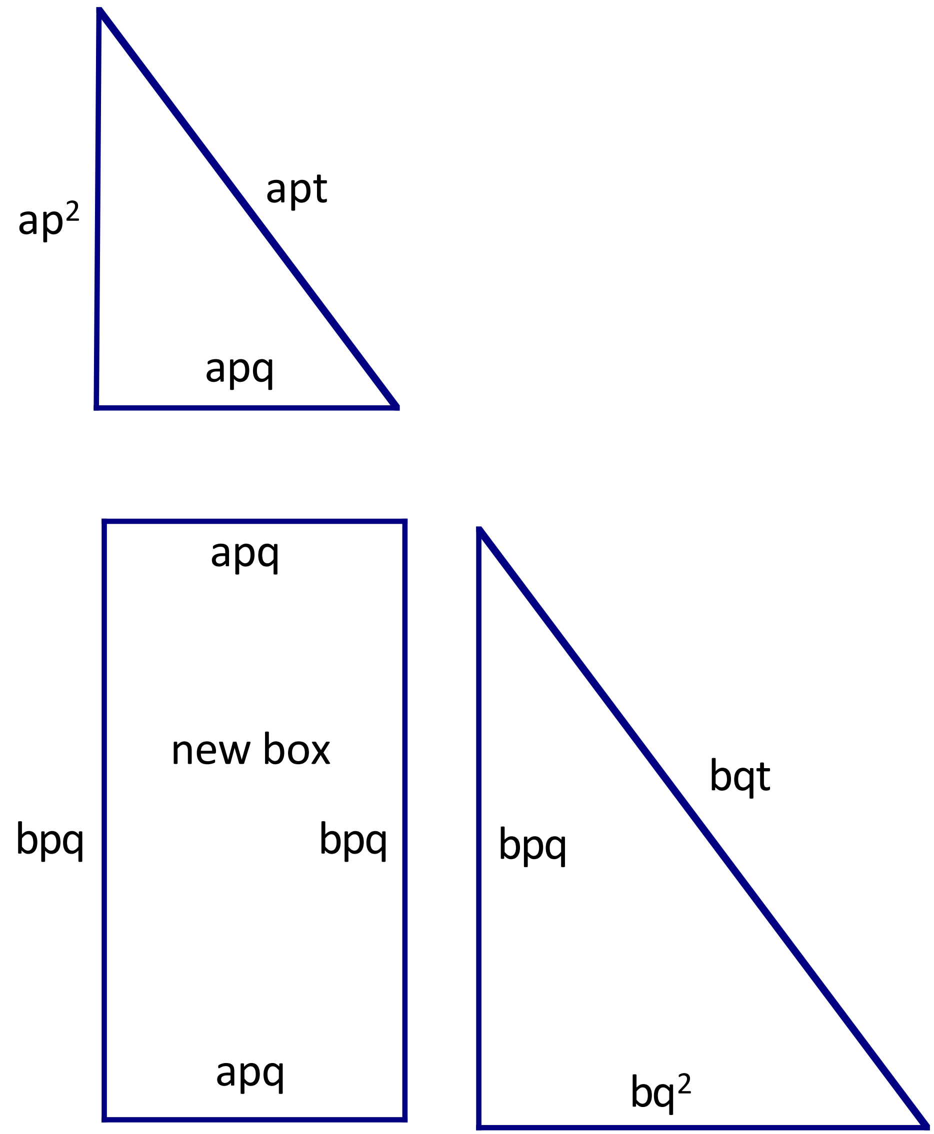
On top of the new box, place a right triangle similar to triangle T. Each side of the new triangle should be multiplied by ap, so the two legs are now ap^2 and apq, and the hypotenuse is apt.



Now place another right triangle similar to triangle T to the right of the box. Each side of this new triangle should be multiplied by bq, so the two legs are bpq and bq^2, and the hypotenuse is bqt.



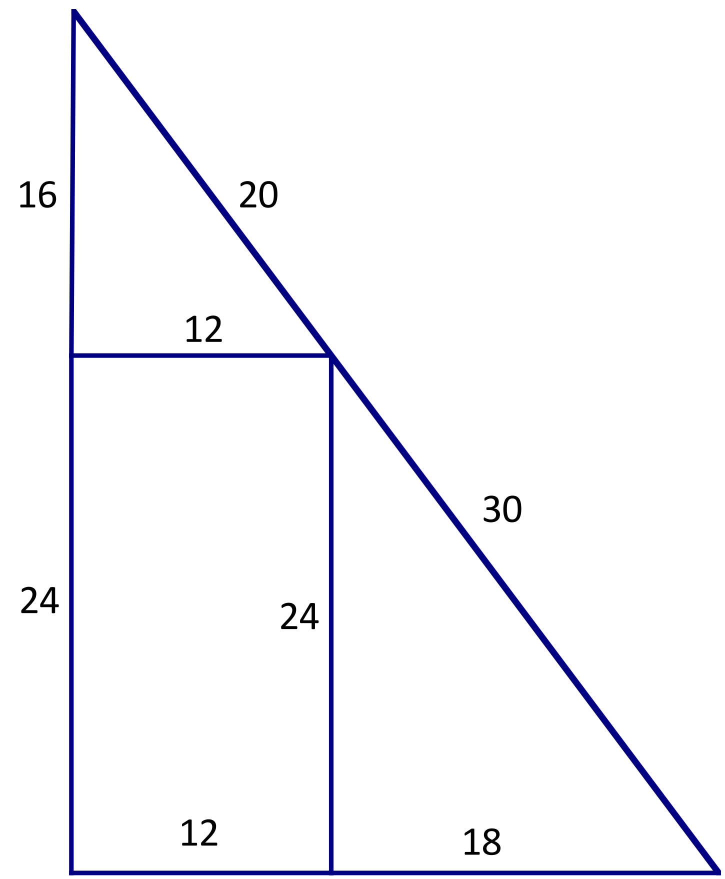
You can see that the length of the ladder is apt+bqt, a whole number; and the height at which the ladder touches the wall is ap^2+bpq, also a whole number. The distance of the bottom of the ladder from the wall is $apq+bq^2$.

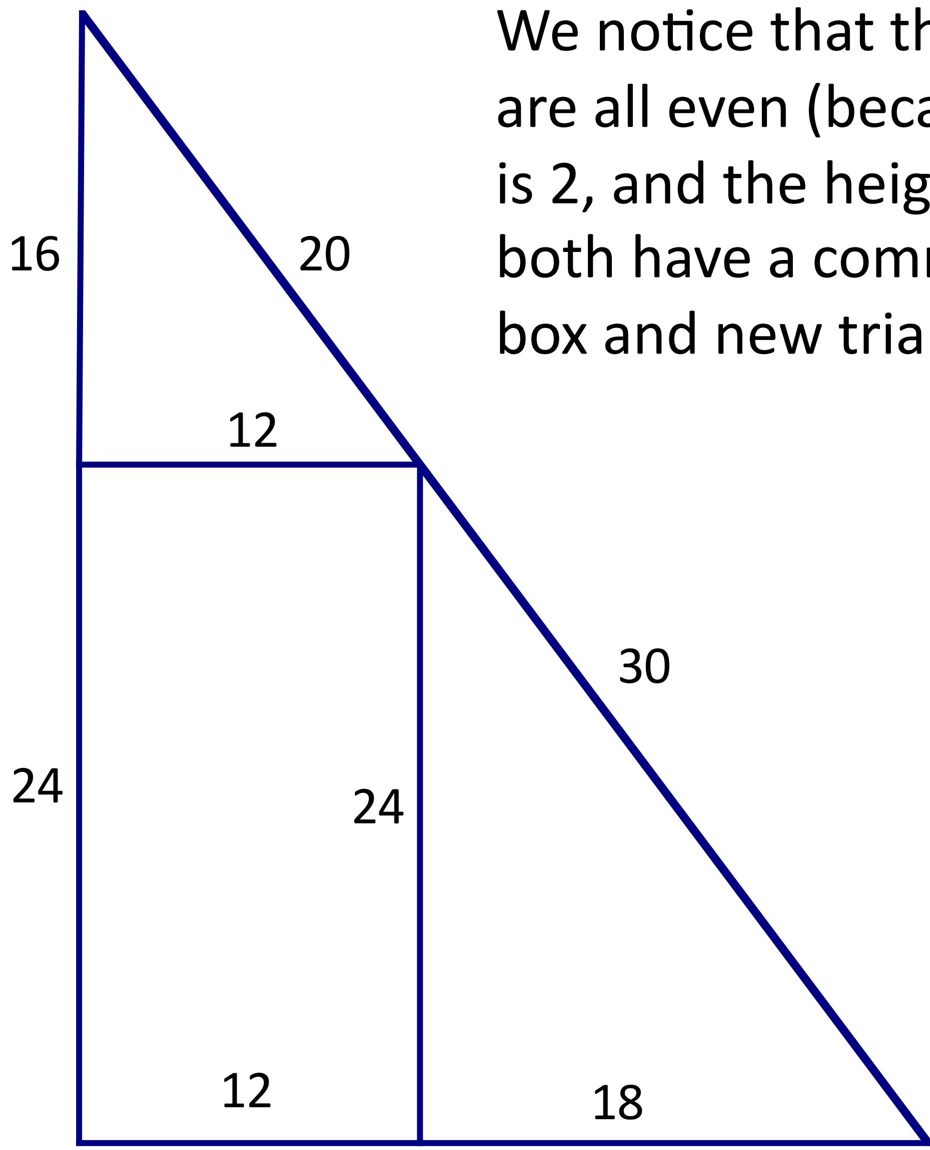


An example

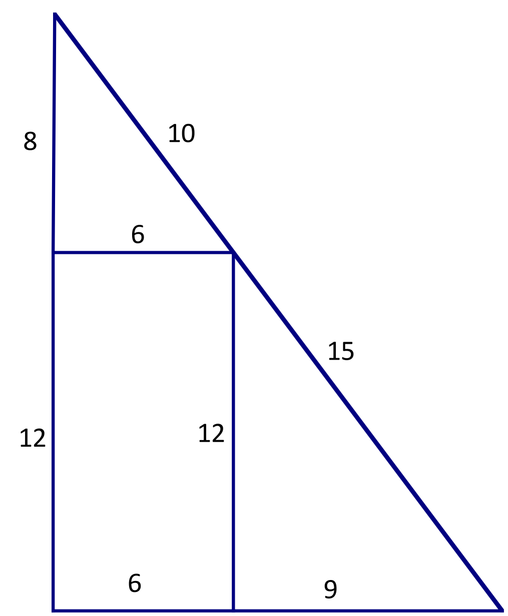
Choose a box with dimensions a=1 and b=2.
Choose a Pythagorean triple triangle, say p=4, q=3, and t=5.
Construct a new box with dimensions apq=12 and bpq=24.
The right triangle on top of the box has legs apq=12 and ap^2=16, and hypotenuse apt=20.
The right triangle on the right side of the box has legs bpq=24 and bq^2=18, and hypotenuse bpt=30.

So the new box and ladder looks as follows:





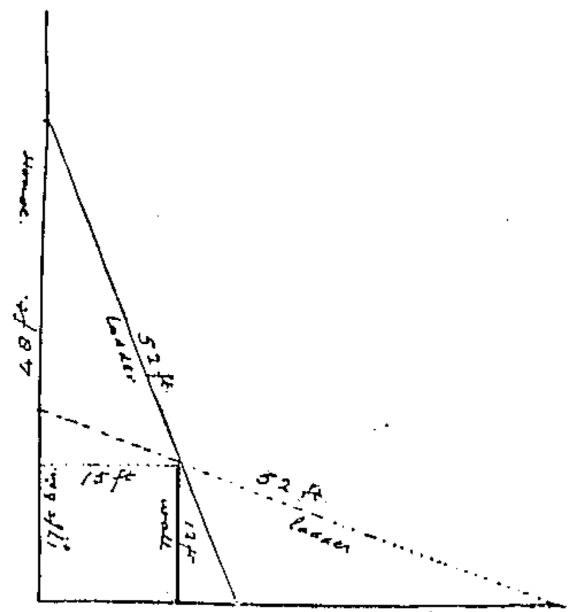
We notice that the numbers in the new triangle are all even (because the height b of the box is 2, and the height p of the triangle is 4, so they both have a common factor of 2). So the new box and new triangle can be scaled down by 2:



Looking again at the example from Pearson (1907):

No. CIII.—CLEARING THE WALL

If a 52-feet ladder is set up so as just to clear a garden wall 12 feet high and 15 feet from the building, it will touch the house 48 feet from the ground.



Our diagram shows this, and also, by a dotted line, the only other possible position in which it could fulfil the conditions, if it were then of any practical use.

The triangle is a 5-12-13 Pythagorean triple: 20-48-52!
For the low-lying ladder, it's a different story: 17.29-49.04-52.

4. Final comments

In current mathematics, the modern meaning of solving a problem is really not much different from the meaning used by Newton and Simpson.

We expect to see a general mathematical procedure to solve some reasonably large class of problems.

And in applied problems we may also require some numerical values of the variables.

The main difference is that the range of problems that can be solved is much larger and the use of technology is getting more and more prevalent.

But the use of technology brings some important changes.

Now a person who solves a problem doesn’t need to know *how* the problem is solved.

So a student who solves the box and ladder problem by writing the equation,

$$b/\sin(x) + a/\cos(x) = s$$

and solves it on a graphing calculator for specific values a, b, and s, does not need to know anything about Newton's method, which provides solutions to these kinds of equations.

A comment about school mathematics

School mathematics is a special case.
(By school math we mean all K-12 math and college math for non-math majors, that is not part of their professional training.)
The concept of solving a problem in school mathematics is just like in recreational mathematics.
Problems are embedded into some narratives (story problems) that are rarely realistic.
What is required is usually just one or a few numbers, and the correctness of the answer is judged by their values.
And also many educational researchers discourage teaching general procedures, and encourage improvisation as being more creative.

This trend goes against the millennia-long trend in the development of mathematics that is moving toward general solutions, which not only provide a numerical answer, but also explain how it can be done, and even why it should be done in this way and not in another way.

But on the other hand, students are still drilled in very specific arithmetic procedures with very narrow ranges of application (for example, addition of common fractions with different denominators) that were designed a hundred years ago for accounting and other practical purposes.

The "ladder and box" problem in modern classrooms

The problem we have described is just one in a group of "ladder" problems (see http://www.mathematische-basteleien.de/ladder.htm#Sliding%20Ladder%20Problems). Others are the "Sliding ladder" as a geometric problem (e.g., Gutenmacher & Vasilyev, 2004, pp. 1-3, 113-114), the "Sliding ladder" as a dynamic calculus problem (e.g., Foerster, 2005, p. 178), the "Shortest ladder" as an optimization problem in calculus (http://sofia.nmsu.edu/~breakingaway/ebookofcalculus), and the "Two crossed ladders" problem (e.g., Gardner, 1979, pp. 62-64; Wells, 1992, p. 131).

With its rich history the “box and ladder” problem can be placed in several strands of high school mathematics.

In geometric constructions done either by hand or with computer software, the problem demonstrates the role of “basic tools”, namely, the class of curves that can be drawn.

In algebra its square box version is a very challenging problem that can be solved by the use of quadratic equations.

Finally, the general problem can be solved easily with calculator technology (see 2. above).
But this presents a dilemma.
Traditionally, solving such problems in school is not a goal in itself.
Instead, it is only done to teach students some techniques and to help them understand more general principles.

Does this mean, for example, that we should not use the TI-84 SOLVER program unless we teach students Newton's method for solving equations, which underlies SOLVER's software?

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