Introducing the notion of variable to young children in courses for elementary teachers using counting boards

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Plan of talk

1. How we have used counting boards in elementary math courses for K-8 preservice teachers

2. What is a variable?

3. The notion of variable in units for young learners
   Some examples

4. The notion of two variables
   Some examples

5. What future teachers studying these units think about them

1. How we have used counting boards in elementary math courses for K-8 preservice teachers

To use a counting board for arithmetic computations, you need a board with an array of numbers with m rows and n columns, some tokens, and some “regrouping” rules for computing with them.
Regrouping is changing the configuration of tokens on a board that preserves the sum of its local values.
We got our idea about how to put numbers on a board from John Napier in his *Rabdology* (1617).

1. How we have used counting boards in elementary math courses for K-8 preservice teachers

To use a counting board for arithmetic computations, you need a board with an array of numbers with m rows and n columns, some tokens, and some "regrouping" rules for computing with them.
Regrouping is changing the configuration of tokens on a board that preserves the sum of its local values.
We got our first idea for a board from John Napier in his *Rabdology* (1617).

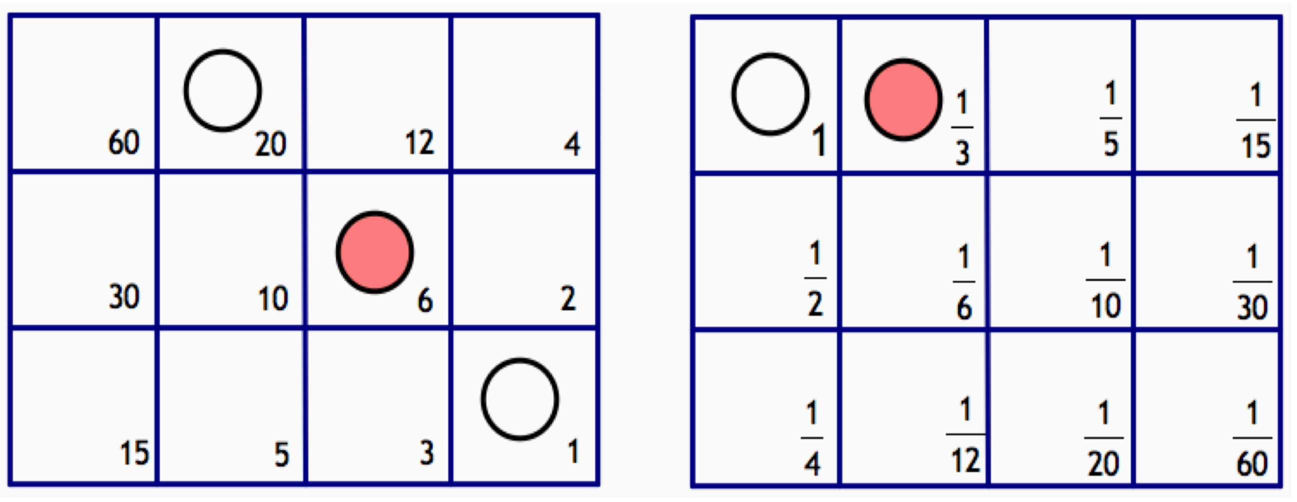
In the boards we use here, the values of locations in any column form a geometric progression with quotient 2. Tokens are of two colors. White tokens have the value 1 and red tokens have the value -1.

Regrouping rules depend on which prime numbers are on the boards.

There are two rules for regrouping tokens in one column:

(1) A token can be exchanged with two tokens that have the same value on the location just below; and
(2) Two tokens of opposite values (1 and -1) can be put on, or removed from, any locations.

There are also between-column regrouping rules that depend on the prime numbers that are involved.



So the three tokens on the left board represent the value 15 = 20 – 6 + 1, and the two tokens on the right board represent the value 2/3 = 1 – 1/3.

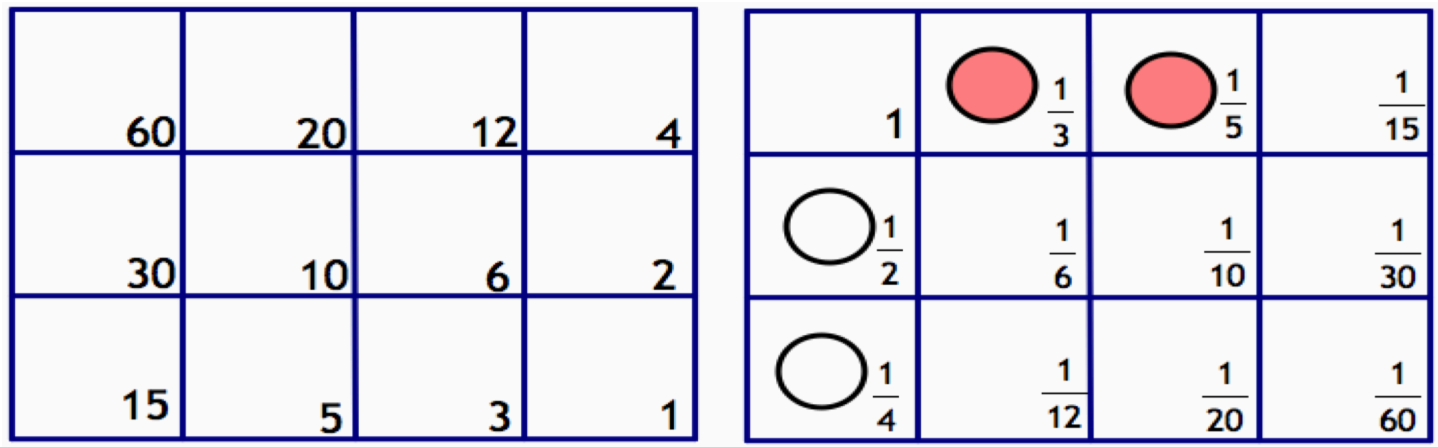
Representing numbers on boards, instead of writing them down, has the following advantages:

1. Students can carry out fairly complex computations before they master “number facts”, by using regrouping rules.

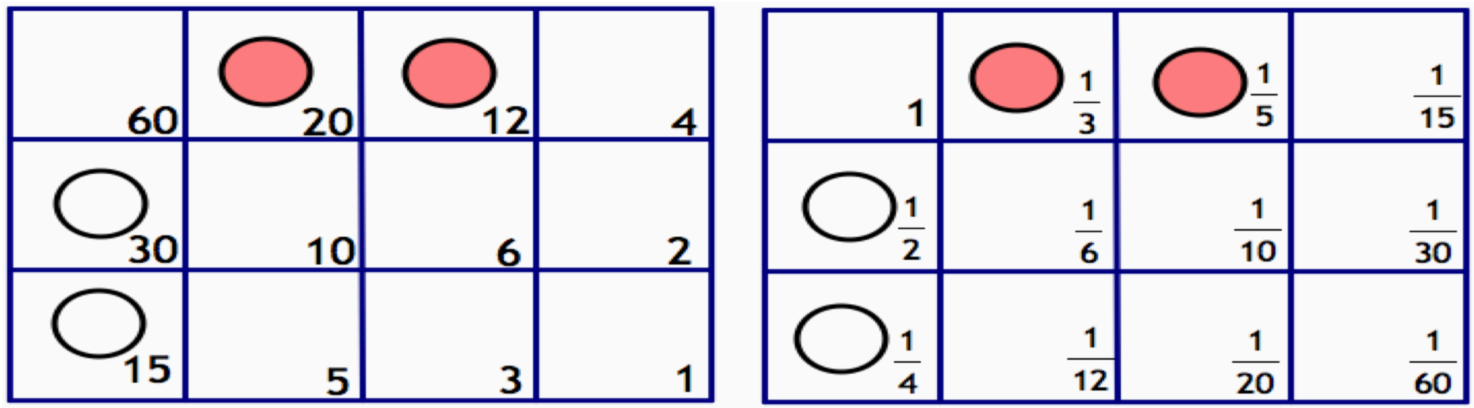
2. Good penmanship is not required.

3. Only very limited knowledge of number-words is necessary. And this feature is important in multi-lingual classrooms.

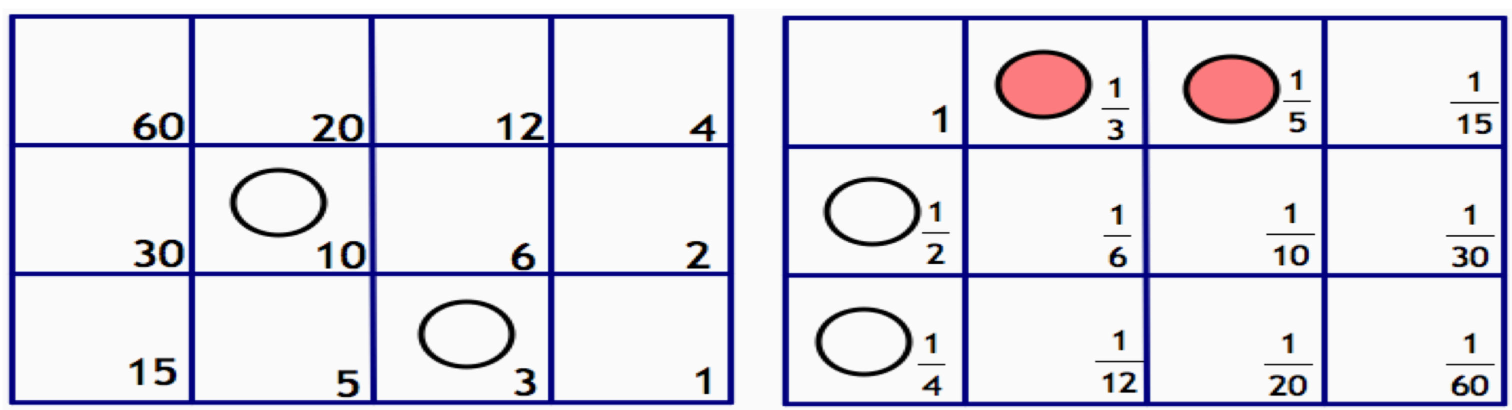
An example: Addition and subtraction of fractions whose denominators are factors of 60



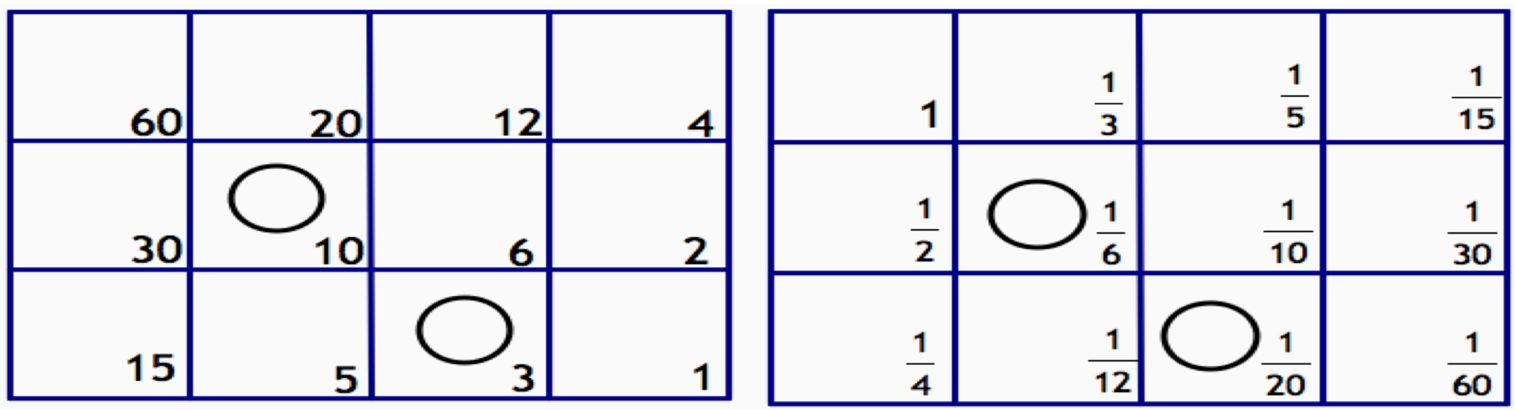
On the fraction board we have a number, ½ - 1/3 + ¼ - 1/5. To compute the total, we put the same configuration of tokens on the integer board.



Now we regroup these tokens, getting 30 – 20 + 15 – 12 =10 + 3 = 13.



We copy these onto the fraction board.



This configuration on the fraction board means that
½ - 1/3 + ¼ - 1/5 = 1/6 + 1/20 = 13/60.
The rules of regrouping are *the same on these two boards* because the values on the integer board are the values on the fraction board multiplied by 60.

2. What is a variable?

Wikipedia:
In elementary mathematics a variable is a symbol, commonly a single letter, that represents a number, called the value of the variable, which is either arbitrary, not fully specified, or unknown…

Making algebraic computations with variables as if they were explicit numbers allows one to solve a range of problems in a single computation.

3. The notion of *variable* in units for early grades.

We use two boards, and we consider here only boards with small whole numbers.
We assume that the values on the two boards are the same, so the two boards form an equation.
(You may imagine that there is an equals sign between the two boards.)

One special token, e.g., a triangular token or a penny, is our variable. It needs a name, such as “triangle” or “penny”, so the students can talk about configurations of tokens. A variable token can have any numerical value. In examples shown here, students start without knowing its value, but compute it from the information provided by a word problem.

When you put a token on the board, the value of the token is the value of the location on which it sits times its value.

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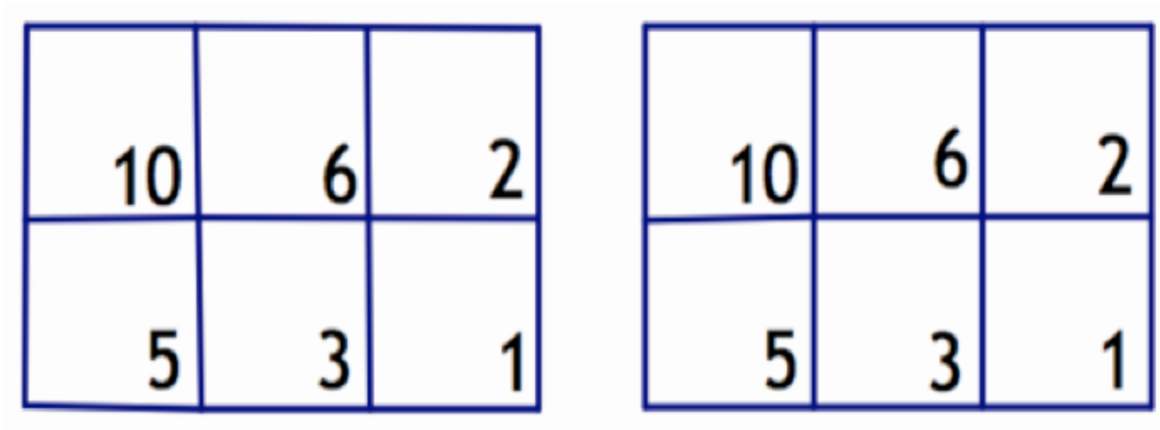
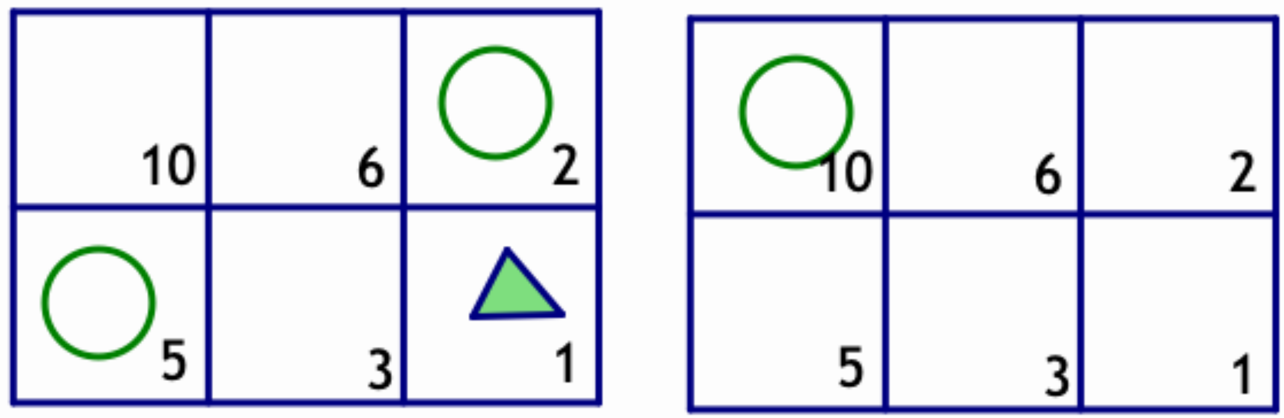
Let’s try a problem with one variable to be solved on two boards.



I'm thinking of a number. When you add 7 to it, you get 10. What is the number?

Using algebra:
Let x equal the number. Then x + 7 = 10
Subtract 7 from both sides: x = 10 – 7 = 3
So the number I'm looking for is 3.

With boards:



My number is the green triangle. The green triangle plus seven equals ten.

The configurations on these two boards replace an algebraic equation.

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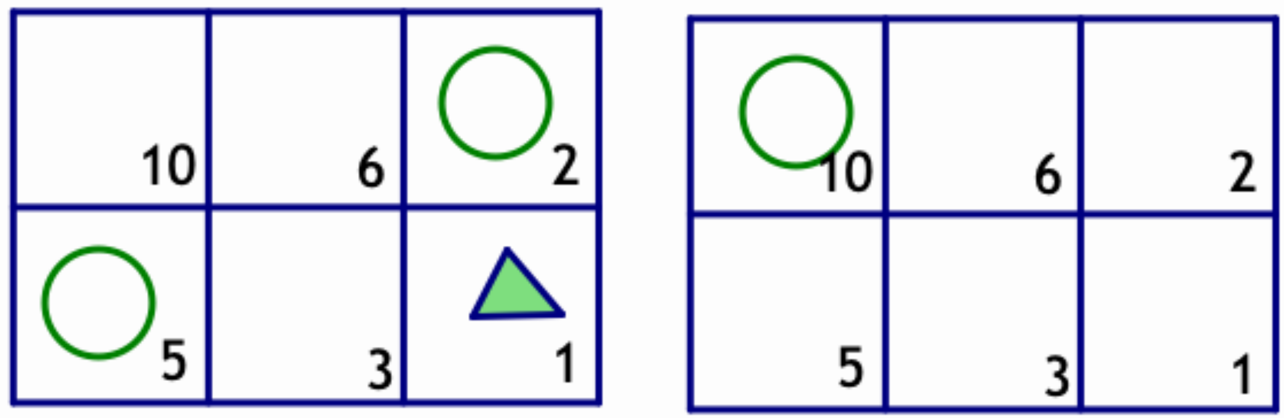

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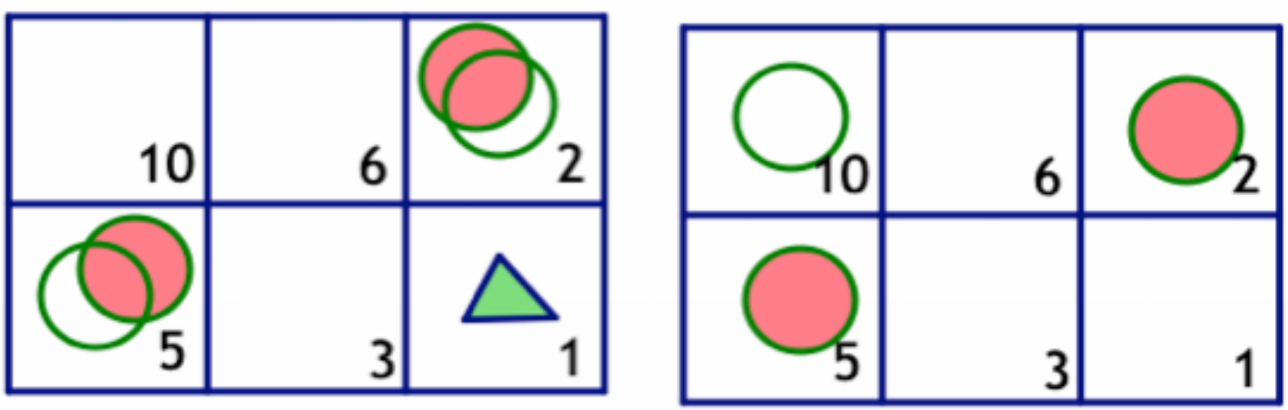
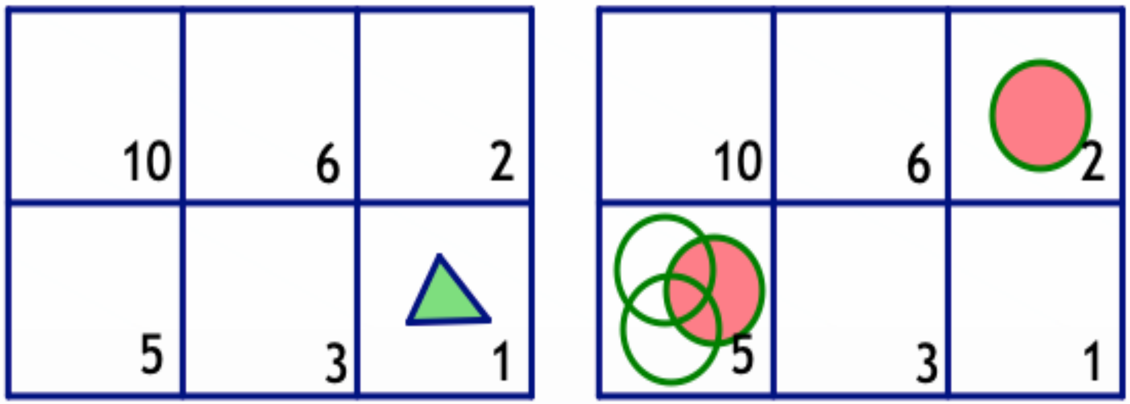
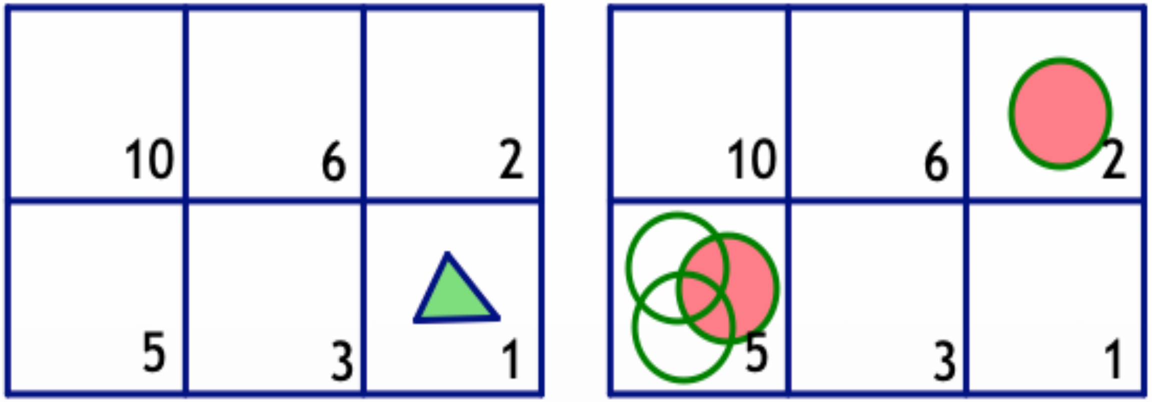


I subtract seven from both sides (both boards).



I regroup 10 as 5 + 5.



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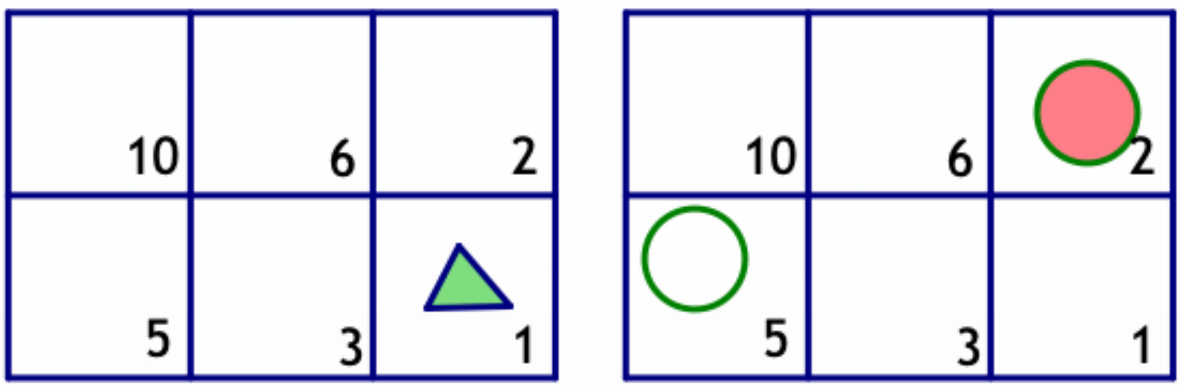
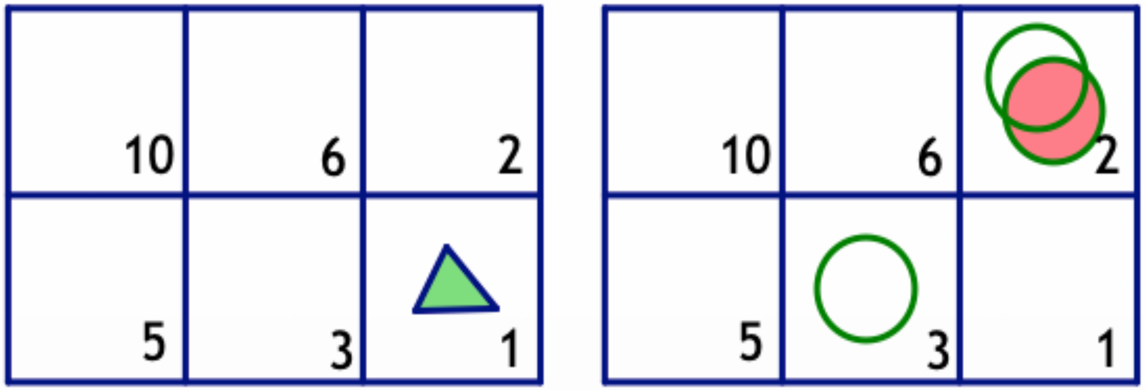
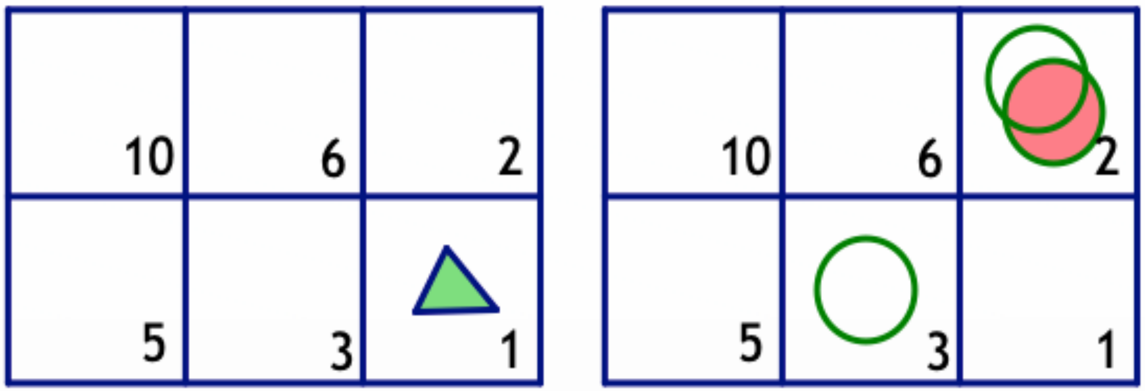


The green triangle equals five minus two.



I regroup 5 as 3 + 2.



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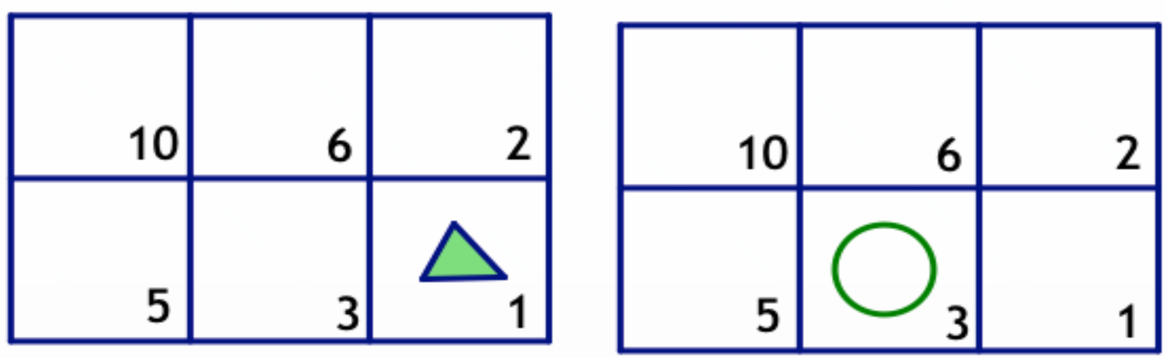


2 minus 2 is zero, so the green triangle equals 3.
Let's check.
I was thinking about 3. When you add seven to it, do you get 10? Yes!

Notice the difference. Here, you solve a problem by manipulating tokens like pieces in a board game instead of by rewriting algebraic expressions.

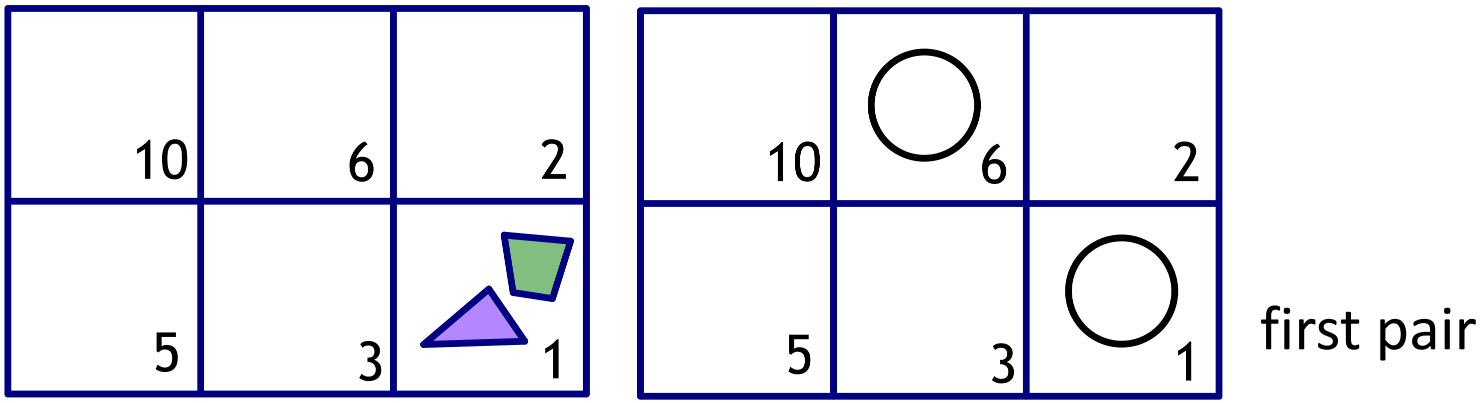
Here are a few more problems that we solved in our class:

- I'm thinking of a number. When you double it and add one, you get 9. What is the number?
- When you subtract 5 from the number I am thinking of, you get 3. What is my number?
- Twice my number plus 4 is 10. What is my number?
- I multiplied my number by 4 and got 24. What is my number?•
- Three times my number plus 4 equals 13. What is my number?
- p + 6 equals 2p - 3. What is p?
- 5p + 4 = 6p – 6. What is p?
- 3p + 10 = 4. What is p?
- 8p – 4 = 10p - 4

The notion of two variables for units in early grades

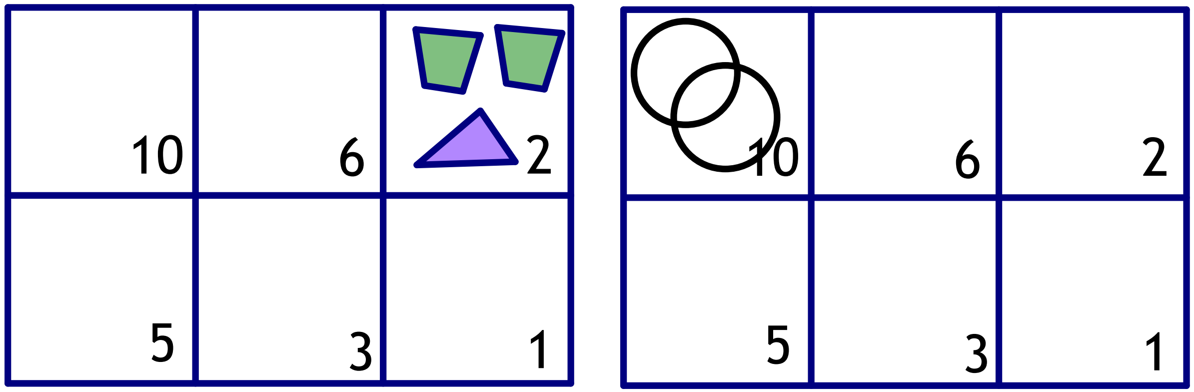
Children and dogs are playing in the park. There are 7 heads and 20 legs in the park. How many children and how many dogs?

We will solve this problem using four boards. Each pair represents one equality.



C + D = 7

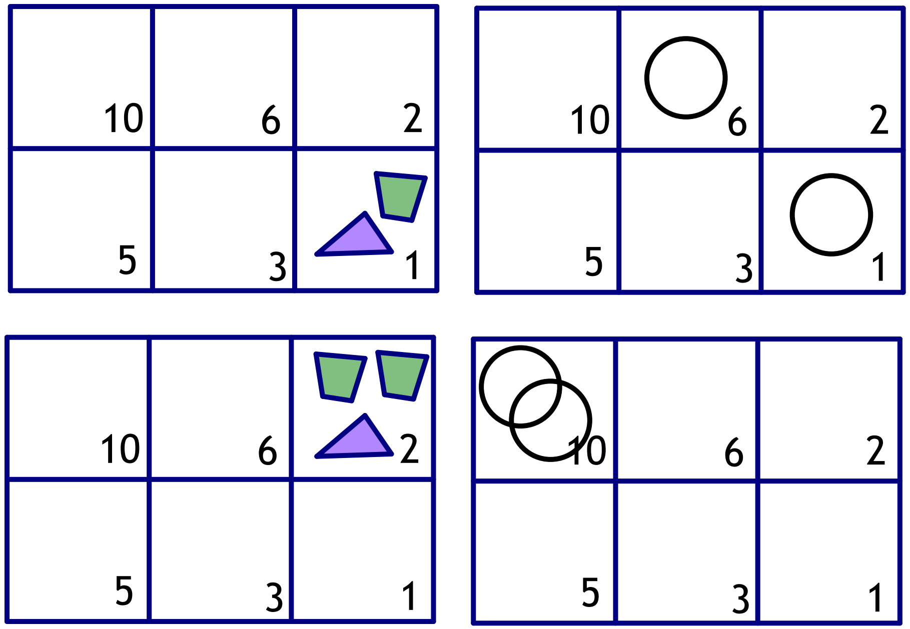
We call the number of children C, and it is the name of the triangle token. We call the number of dogs D, and it is the name of the green token. As before, we call these tokens “variables”, because we will find their values when we solve the problem. The value on the left board above is C + D, and the value of the right board is 7. We read the two boards together as the equation C + D = 7.



second pair

2C + 4D = 20

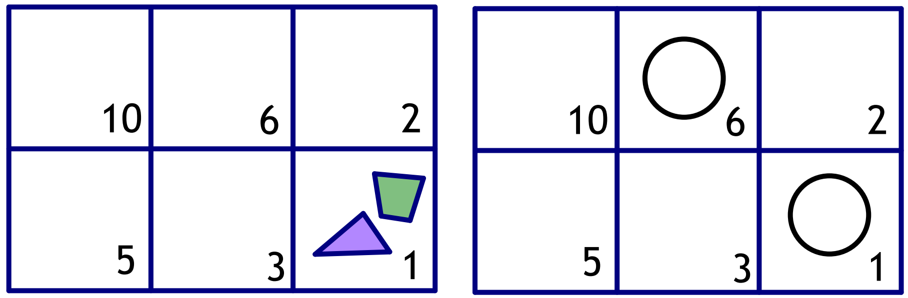
Children have 2 legs and dogs have 4 legs, and the total number of legs is 20. So the two boards in the second pair represent 2C + 4D = 20.



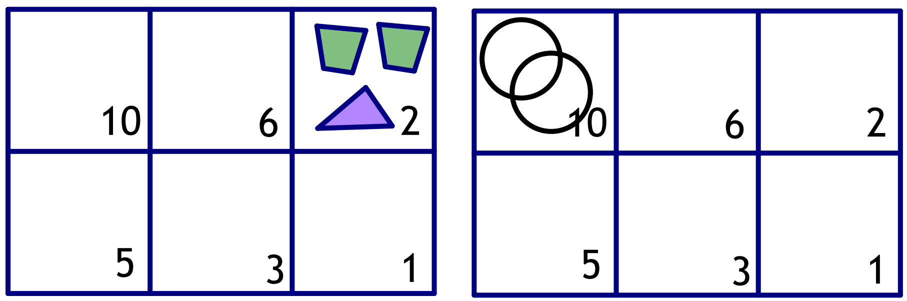
C + D = 7
first pair

2C + 4D = 20
second pair

Now we have to get rid of one of the two variables on either the first or the second pair of boards.

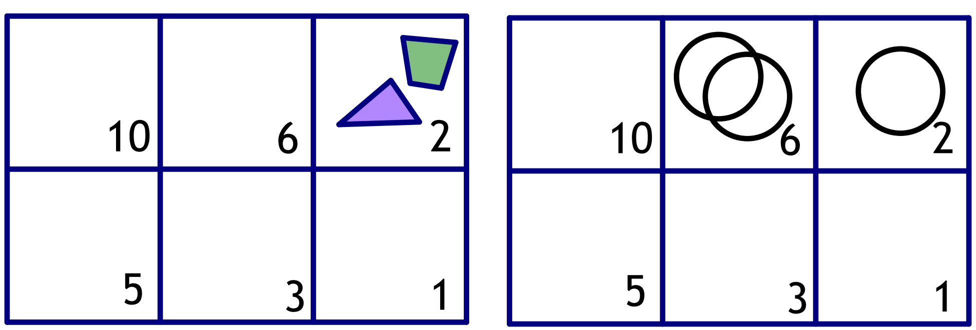


$C + D = 7$
first pair



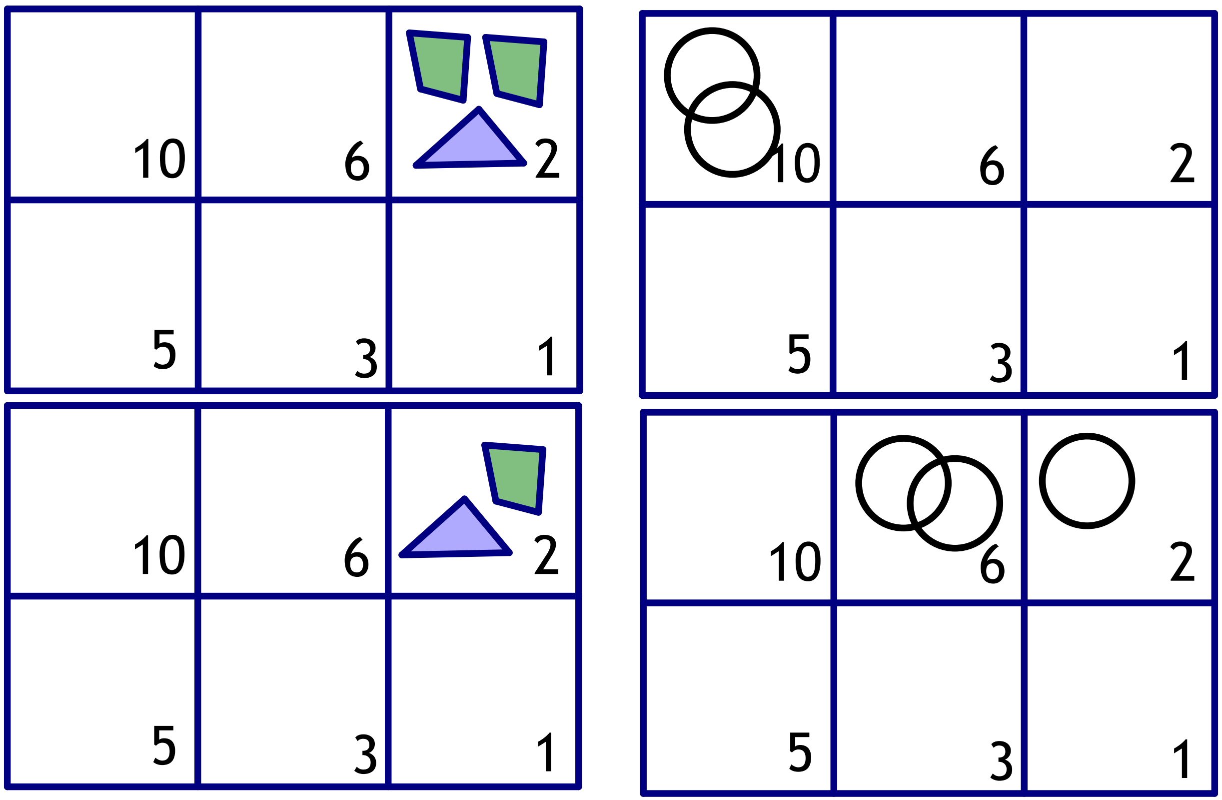
$2C + 4D = 20$
second pair

Now we have to get rid of one of the two variables on either the first or the second pair of boards.
First let's double the values on the first pair of boards.



$2C + 2D = 14$
first pair

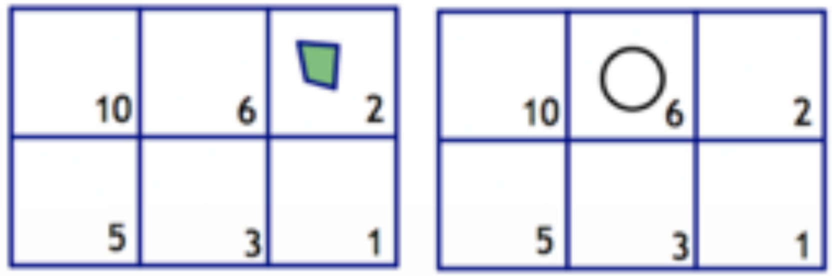
Let’s put the two pairs of boards together. We want to subtract the values on the lower boards from those on the upper boards.



2C + 4D = 20
second pair

2C + 2D = 14
first pair

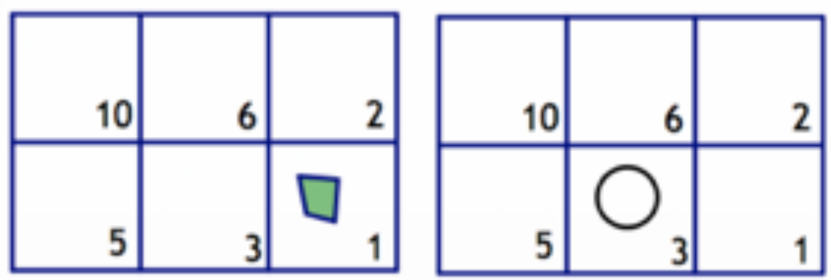
When we subtract the values shown on the first pair of boards from the values on the second pair, we can get rid of 2C and have only 2D remaining on the left board (and 20 – 12 – 2 = 6 remaining on the right board):



2D = 6

2D = 6
second pair

Two times the number of dogs equals 6. We divide by 2 by moving the counters one row below on the boards.

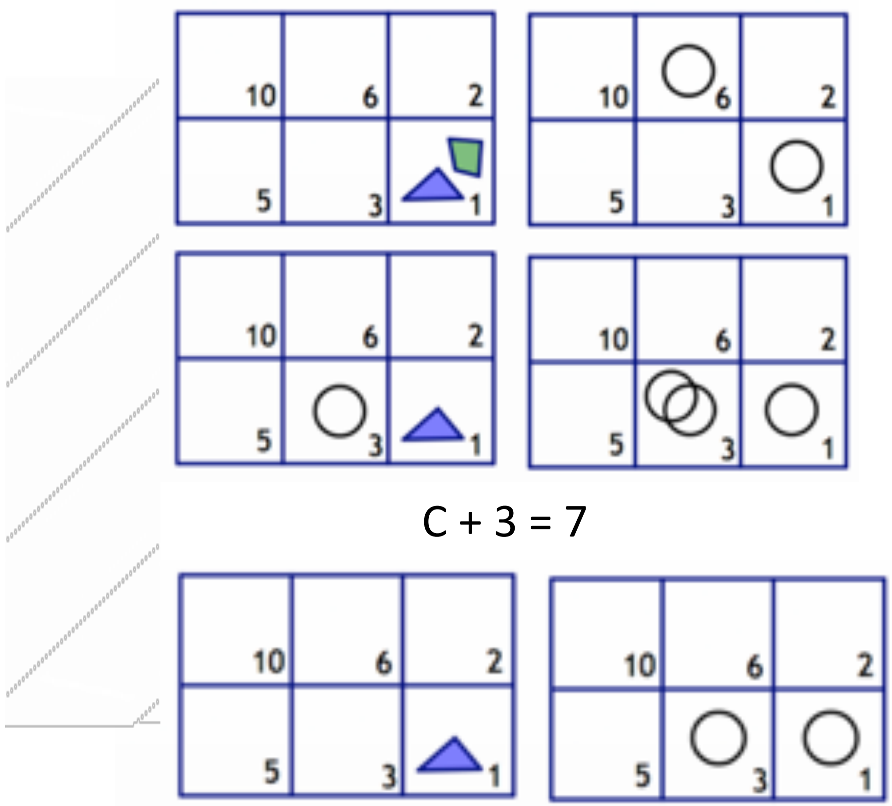


D = 3

D = 3
second pair

So the number of dogs is 3.

We return to our first pair and replace the green token by a white token on location 3 , and then we regroup.



C + D = 7

C + 3 = 7

C = 7 – 3
C = 4

We subtract 3 from both boards. The value of the purple triangle is 4. There are 4 children!

Let's check:

3 dogs plus 4 children = 7 playing in the park

4 children and 2 legs per child = 8 legs

3 dogs and 4 legs per dog = 12 legs

8 legs + 12 legs = 20 legs.

We got it!

Two more problems on our website:

The sum of the two digits of a two-digit number is 11. The tens digit is one more than 4 times the units digit. What are the two digits?

Two farmers, Fred and Violet, were taking their sheep to market. Fred said, "Give me one of your sheep so we will both have the same amount." Violet answered, "No! Instead, you better give me one of yours, so I'll have twice as many as you have.“

https://web.nmsu.edu/~pbaggett/NumberBoards/index.html

5. What future teachers in my courses think about these units

In Spring 2019 we occasionally used counting boards in two elementary math classes for prospective (and a few practicing) K-8 teachers.

In an anonymous questionnaire at the end of the course (before grades were posted; also the instructor saw the questionnaires after the grades were posted), the following was asked:

We did some units with counting boards, e.g., I'm thinking of a number, and word problems such as Heads and Legs and Two farmers, Fred and Violet. Did you find them useful? Why or why not? I am really interested in your opinion about them. Let me know.

Here are responses from 13 students.

- No, I just don't think I will use these in my classroom.
- No. I personally like algebra better. I understand for kids they need visuals, but I prefer numbers.
- No. They were hard to follow and confused me more than helped me.

- I believe the class had mixed feelings about this. However I enjoyed it because it was different. Challenging at first, but I understood it with the problems given.

- I did find them useful. They are very different so I assume they would be very engaging (for children).
- Yes, because it's doing algebra in a way not thought of.

- Yes, I think knowing different ways to use the counting boards was really useful.
- Yes, Heads and legs was one of the best/most fun.
- Yes, not only did it help me as well but it will help my students in the future.
- I found them useful. It helped me see problems in a more simple way.
- They truly were amazing lessons that will help children (I know) understand concepts behind them.
- Word problems were my favorite! Yes, it helps kids connect to math on a new live (active) level.
- They were very useful. I really liked the new age approach to old school math. I think children will enjoy and understand this better.

Thank you!

For more units using counting boards,
https://web.nmsu.edu/~pbaggett/NumberBoards/index.html