Comparing two first-year algebra books from the 1840's, Warren Colburn's *An Introduction to Algebra* and Joseph Ray's *Algebra: Part First*, to today's "modern" first-year algebra curriculum

2018 Joint Mathematics Meetings San Diego, CA
Session on History or Philosophy of Mathematics
Room 28D, Upper Level, San Diego Convention Center
Thurs. Jan. 11, 8:30 AM

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Plan of talk

1. The main sources examined for this presentation

2. A brief comparison of their contents
    a. Quantity, number, variable
    b. Negative quantities in Colburn and Ray
    c. Imaginary quantities and complex numbers

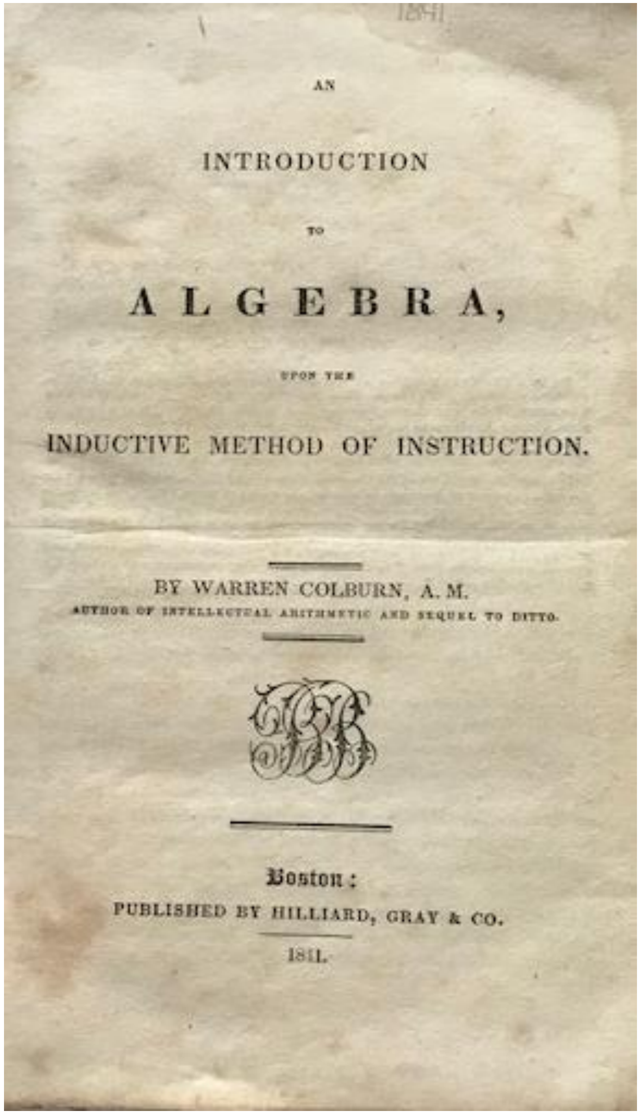
3. Final remarks

1. The main sources examined for this presentation

- Warren Colburn's 1841 *An Introduction to Algebra upon the Inductive Method of Instruction*

- Joseph Ray's 1848 *Algebra Part First: on the Analytic and Inductive Methods of Instruction*

- Common Core Mathematics Standards: High School: Algebra
- *Algebra 1 Common Core*, 2012 (8 authors, Pearson Publishing)
- *Algebra I A Common Core Math Program* Student Text Volume 1
  2012 (8 authors, Carnegie Learning)


AN
INTRODUCTION
TO
ALGEBRA,
UPON THE
INDUCTIVE METHOD OF INSTRUCTION.

BY WARREN COLBURN, A. M.
AUTHOR OF INTELLECTUAL ARITHMETIC AND SEQUEL TO DITTO.

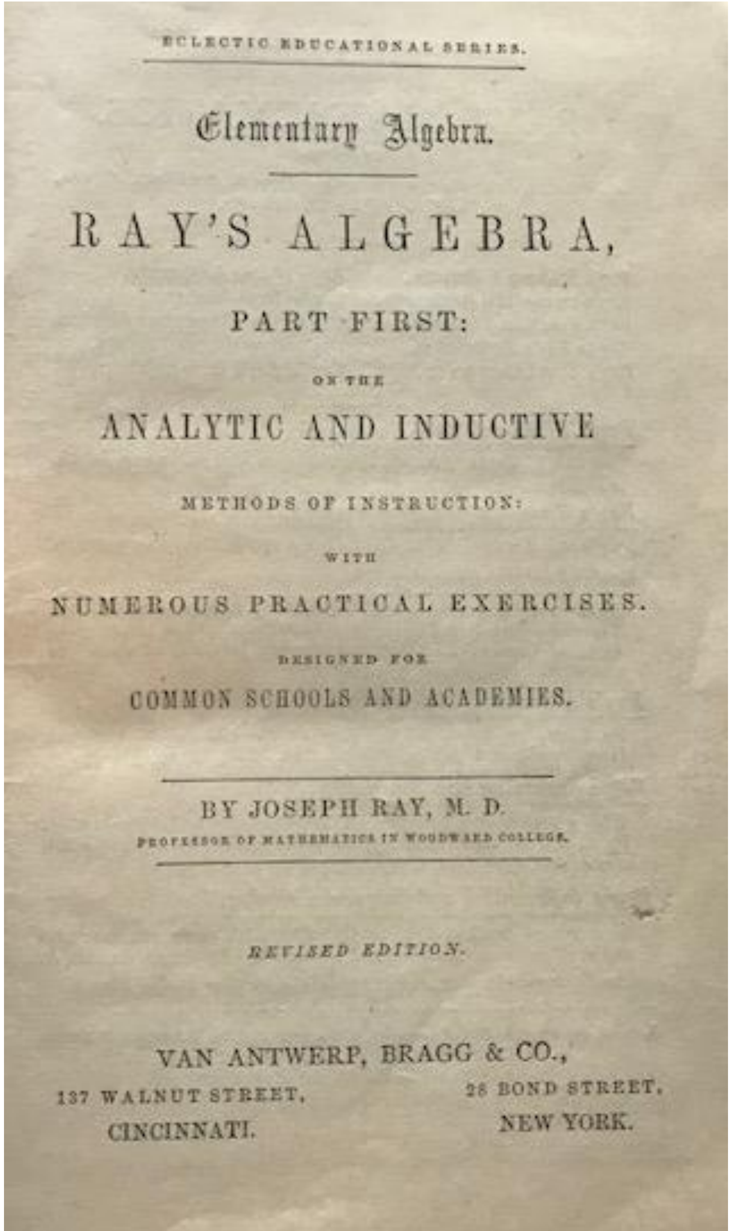
Boston:
PUBLISHED BY HILLIARD, GRAY & CO.
1841.

An Introduction to Algebra upon the Inductive Method of Instruction (1825; 1841) by Warren Colburn (1793-1833)

276 pages, 5 by 7.5 inches

Colburn graduated from Harvard College in 1823.

He is best known for his *First Lessons in Arithmetic on the Plan of Pestalozzi* (1822), which sold ~2 million copies.



Ray's Algebra, Part First: on the Analytic and Inductive Methods of Instruction (1848)
by Joseph Ray, M.D. (1807-1855)

249 pages, 5 x 7.5 inches

Ray became Professor of Mathematics at Woodward College in Cincinnati in 1836.

"Joseph Ray did for figures what McGuffey did for literature."

More than 50 titles, including revised editions, appeared in Ray's Mathematical Series. The core of the series consisted of 6 books: *Primary, Intellectual, Practical*, and *Higher Arithmetic*, and *Elementary* and *Higher Algebra.*

ECLECTIC EDUCATIONAL SERIES.

Elementary Algebra.

RAY'S ALGEBRA,

PART FIRST:

ON THE

ANALYTIC AND INDUCTIVE

METHODS OF INSTRUCTION:

WITH

NUMEROUS PRACTICAL EXERCISES.

DESIGNED FOR

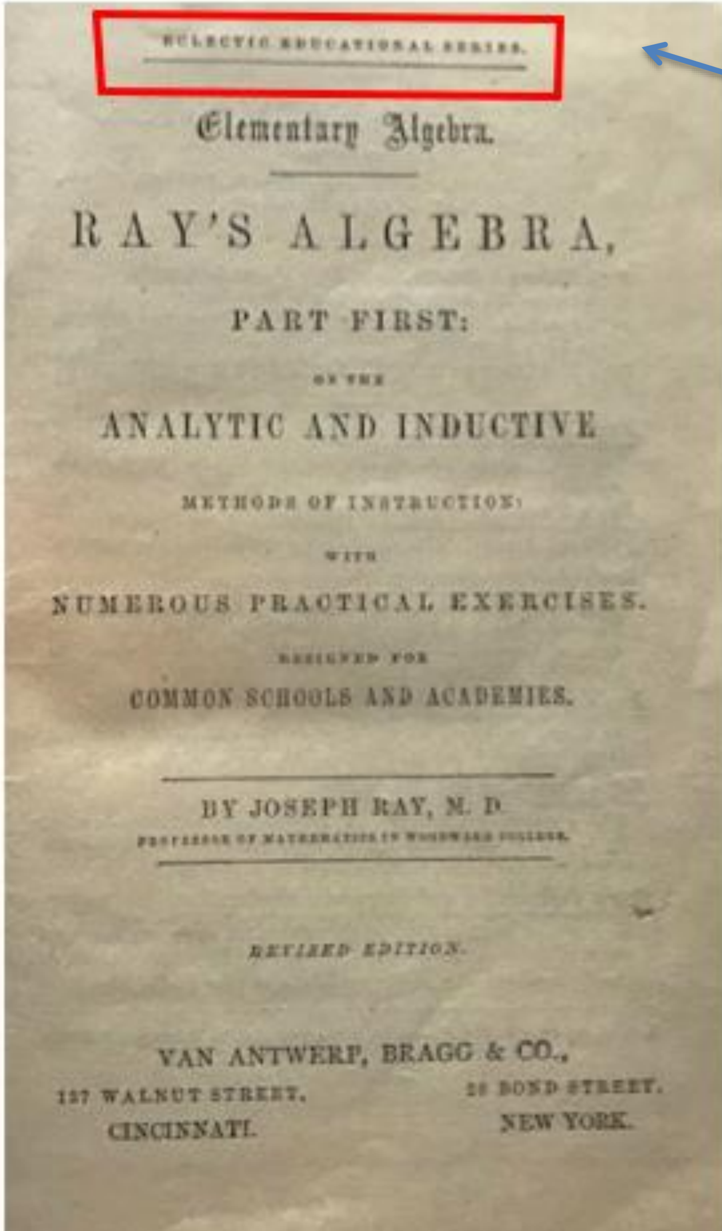
COMMON SCHOOLS AND ACADEMIES.

BY JOSEPH RAY, M. D.
PROFESSOR OF MATHEMATICS IN WOODWARD COLLEGE.

REVISED EDITION.

VAN ANTWERP, BRAGG & CO.,
137 WALNUT STREET, CINCINNATI.
28 BOND STREET, NEW YORK.

ECLECTIC EDUCATIONAL SERIES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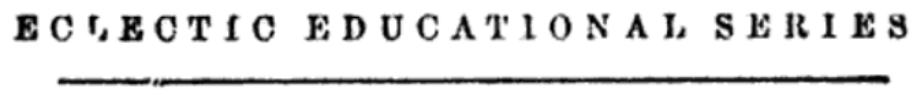
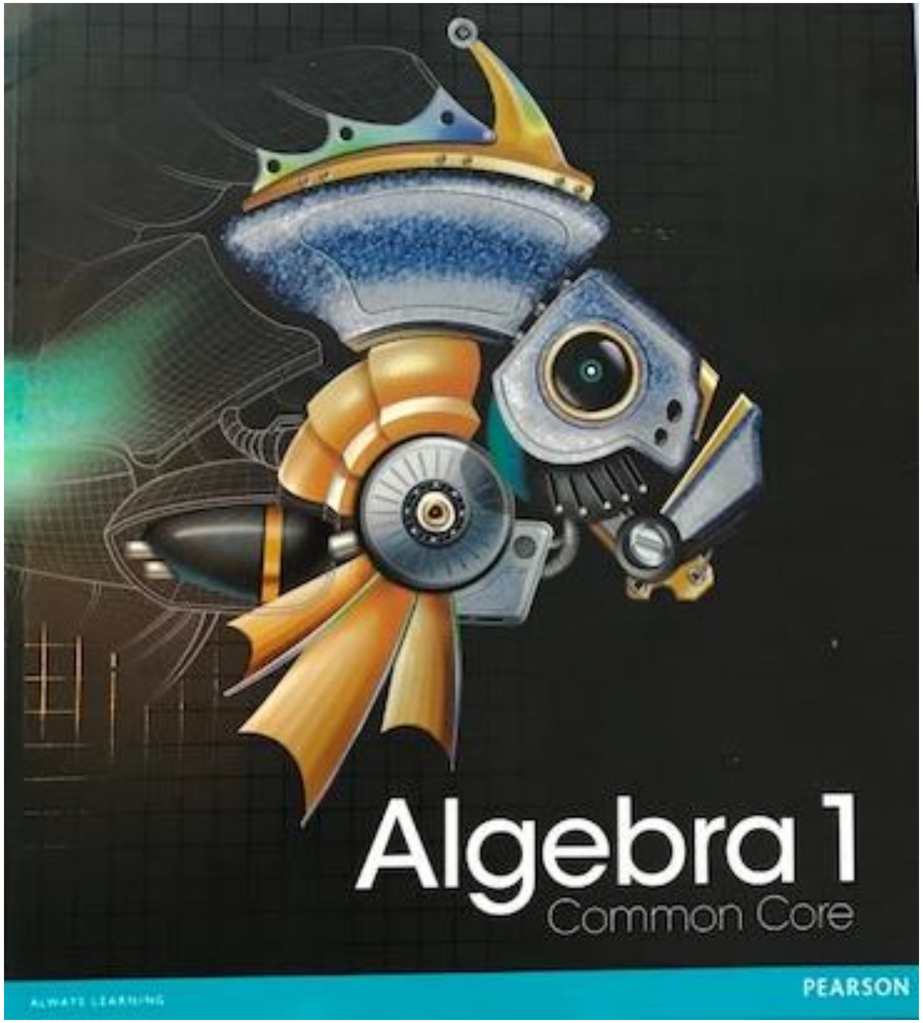
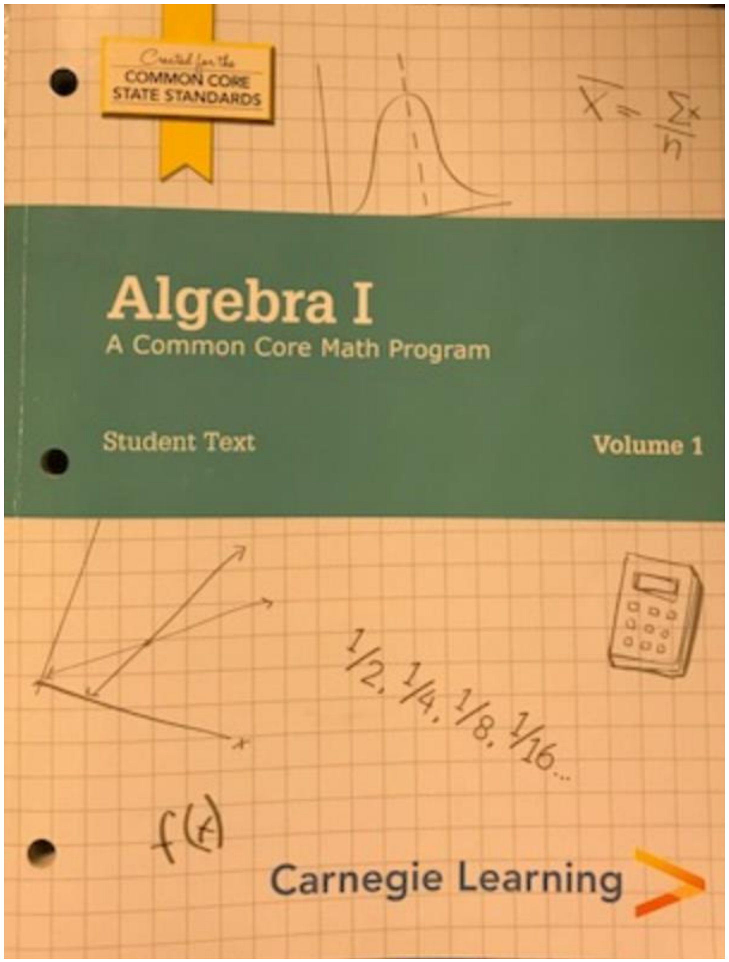


Books in the Eclectic Educational Series are used today by the Home Schooling Association. Ray's *Algebra* is one such book.



Algebra 1 Common Core
(2012)
Eight authors
946 pages, 9 by 10 inches

"Math is a powerful tool with far-reaching applications throughout your life. We have designed a unique and engaging program that will enable you to tap into the power of mathematics and mathematical reasoning."



Algebra I
A Common Core Math Program
(2012) Eight authors
500 pages, 8.5 by 11 inches

"You are about to begin an exciting endeavor using mathematics! To be successful, you will need the right tools. This book is one of the most important tools you will use this year. … You will be given opportunities to … use tools such as tables, graphs, and graphing calculators."

2. A brief comparison of their contents

a. Quantity, Number, and Variable

Text.

Colburn, p. 10. In questions proposed for solution, it is always required to find one or more **quantities** which are unknown; these, when found, are answers to the question.

Ray, p. 25. In Algebra, **numbers** and **quantities** are represented by symbols, the letters of the alphabet. **Quantity** is anything that is capable of increase or decrease; such as numbers, lines, space, time, motion, &c….
The **numerical value** of any **quantity** is the number that expresses how many times it contains its unit of measure.
Number is an expression denoting a unit or a collection of units. Numbers are either abstract or concrete.

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Comment.

For Colburn and Ray:
A quantity is anything that can be measured: length, weight, space, as well as number, etc. And it can be unknown.
The **numerical value** of a quantity covers what we today call integers and rational and real numbers.
Numbers measure quantities.
Number is (usually) a whole number, but Colburn says later in his book that it can also be a fraction.

Text.

Common Core Standards

Use **variables** to represent **quantities** in a real-world or mathematical problem; and construct simple equations and inequalities to solve problems by reasoning about the **quantities**.

Algebra 1 Common Core 2012, p. 4

A mathematical **quantity** is anything that can be measured or counted. Some **quantities** remain constant. Others change, or vary, and are called **variable quantities**. Example: A dozen is another way to describe a quantity of 12 eggs.

Algebra uses symbols to represent **quantities** that are unknown or that vary….
A **variable** is a symbol, usually a letter, that represents the value(s) of a **variable quantity**. An algebraic expression is a mathematical phrase that includes one or more **variables**. A numerical expression is a mathematical phrase involving numbers and operation symbols, but no **variables**.

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Comment.

The word **variable** does not occur in Colburn or in Ray.
The modern term **variable quantity** has the same meaning as **quantity or unknown quantity**, in Colburn and Ray.
And a **variable** in the Common Core is just a written symbol.

b. Negative quantities in Colburn and Ray

Text.

Colburn, p. 114

As these negative quantities may frequently occur, it is necessary to find rules for using them.

In the first place, let us observe that all **negative quantities** are derived from endeavoring to subtract a larger quantity from a smaller one. **The largest number that can** actually **be subtracted** from any number, **is the number itself**. Thus the largest number that can be subtracted from 5 is 5; the largest number that can be subtracted from *a* is *a* itself. **If it be required to subtract 8 from 5, it becomes 5 – 5 – 3 = -3; the 5 only can be subtracted, the 3 remains with the sign -, which shows that it could not be subtracted**. If 5 be subtracted from 8, the remainder is 3, the same as in the other case except the sign.

... Hence, adding a negative quantity, is equivalent to subtracting an equal positive quantity.

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Comment.

In Colburn,
the **negative quantity** -3 is the **number** 3 waiting to be **subtracted**.

Text.

Ray, p. 44

Some algebraists say, that numbers with a positive sign represent quantities greater than 0, while those with a **negative sign**, such as -3, **represent quantities less than nothing**. The phrase, **less than nothing**, however, **can not convey an intelligible idea**, with any signification that would be attached to it in the ordinary use of language…

The idea would be properly expressed by saying that **negative quantities are relatively less than zero**.

Thus, if we take any number, for instance 10, and add to it the numbers 3, 2, 1, 0, -1, -2, and -3, we see that adding a negative number produces a less result than adding zero.

10	10	10	10	10	10	10
3	2	1	0	-1	-2	-3
13	12	11	10	9	8	7

Text.   .

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From this, we also see that adding a negative number produces the same result as subtracting an equal positive number.

Comment.

Ray says that ***less than nothing*** is **not an intelligible idea**. He suggests that **negative quantities** should be referred to as ***relatively*** **less than zero**.

And he concludes that adding a negative number produces the same result as subtracting an equal positive number.

Colburn and Ray do not treat **numbers** and **negative quantities** as being parts of the same number system that is now called **integers**.

Text.

Common Core

Some equations have no solutions in a **given number system**, but have a solution in a **larger system**.
For example, the solution of $x + 1 = 0$ is an **integer, not a whole number**;
the solution of $2x + 1 = 0$ is a **rational number, not an integer**;
the solutions of $x^2 – 2 = 0$ are **real numbers**, **not rational numbers**;
and the solutions of $x^2 + 2 = 0$ are **complex numbers**, **not real numbers**.

… Show that a **number** and its **opposite** have a sum of 0 (are additive inverses).

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In the Common Core, there are not just **numbers**, but different **number systems**.

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In the Common Core, there are not just **numbers**, but different **number systems**.

The **negative quantities** in Common Core are introduced as **opposites** of positive **numbers.**
The text says that adding a **number** and its **opposite** cancels. (How it relates to subtraction is not specified.)

c. Imaginary quantities and complex numbers

Text.

Colburn, p. 179

If q is greater than $p^2/4$, the quantity $(p^2/4 - q)^{½}$ becomes negative, and the extraction of the root **cannot be performed**. The **values** are then said to be **imaginary**.

Ray, p. 185.

$\surd$-9, $\surd$-4a², $\surd$-b are algebraic symbols, which indicate **impossible operations**. Such expressions are termed **imaginary quantities**. They occur, in attempting to find the value of the unknown quantity in an equation of the second degree, where some absurdity or impossibility exists in the equation, or in the problem from which it was derived.

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Comment.

In Colburn and Ray, imaginary values are the result of operations which **cannot be performed**, namely, they are **values that can be imagined but which don't exist**.

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Common Core

Represent **complex numbers** on **the complex plane in rectangular and polar form** (including **real and imaginary numbers**).

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Comment.

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Common Core requires that students learn two **geometric representations** of **complex numbers**, and that such representations explain what those numbers are.

3. Final remarks

Overall differences in the 1840's books and modern math

a. Algebra changed.

Concepts such as real numbers, complex numbers, and the modern concept of function, were simply not present. Historian Ivor Gratton-Guinness (1997) said that "Algebra" changed into "Algebras".

In the 1840's there were two views of early algebra, algebra as computing with letters, and algebra as generalized arithmetic.
But algebra in the 1840's was still one domain.

Today there are many different algebraic systems. The rules for each are different. And in today's school algebra there is an attempt to cover as many subdomains of algebra as possible.

b. Early algebra was separated from geometry. And today, geometry, both analytical and informal, is extensively used in teaching algebra.

c. The concepts of functions and their graphs were not present.

d. The use of technology is an important part of the Common Core.

The content of the books from the 1840’s, especially Ray, which is used by home-schoolers today, is significantly different from the content of the Common Core algebra books.

There is a claim used to justify teaching algebra to all students:

“Learning algebra is necessary in order for one to proceed in learning higher math.”

The content of the books from the 1840’s, especially Ray, which is used by home-schoolers today, is significantly different from the content of the Common Core algebra books.

There is a claim used to justify teaching algebra to all students:

“Learning algebra is necessary in order for one to proceed in learning higher math.”

How is it possible that two such different algebra texts can be a prerequisite for future learning?

Maybe the claim that algebra must be a prerequisite is false?

Thank you

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https://www.math.nmsu.edu/~breakingaway/

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Charles, R., Hall, B., etc. (2012). *Algebra 1 Common Core*. Boston: Pearson.

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Common Core Mathematics Standards: High School: Algebrahttp://www.corestandards.org/Math/Content/HSA/introduction/

Eclectic Educational Series, Home Schooling Association: http://www.dollarhomeschool.com/raysarithmetic/index.html

Grattan-Guinness, Ivor (1997). *The Rainbow of Mathematics: A History of the Mathematical Sciences*. New York: W.W. Norton, p. 4.

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Ray, Joseph (1866). *Ray's Algebra, First Book: Elements of Algebra*. Cincinnati, OH: Wilson Hinkle & Co.