A Course in the History of Mathematics Education

Session on
Relations between the History and Pedagogy of Mathematics
American Mathematical Society Central Section Meeting
University of Wisconsin
Madison, WI
10 AM Saturday, September 14, 2019

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Plan of talk

- Brief description of the graduate mathematics course

- Works of four students from the most recent class

    - An 1870 arithmetic text in the Cherokee language
    - Is *Number Stories of Long Ago* (1915) by David Eugene Smith ethnomathematics?
    - Comparing *Geometria Plana y del Espacio* (1915) and *Plane and Solid Geometry* (1913), both by G.A. Wentworth and David Eugene Smith
    - A story problem about sharing bread: Its 7th century Arabic origin and its appearance in other locations

- Final remarks

- Brief description of the graduate mathematics course

From the NMSU Course Catalog

A study of the history of the mathematics taught in American schools, including an examination of authentic original textbooks and the changes in their content and the approach to the subject over time, together with writings of people who have influenced the development and changes of mathematics education. Theories of learning mathematics, and current issues in mathematics education.

- Brief description of the graduate mathematics course

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Three parts: Readings, Book reports, Final projects

*Readings

passed out one session and discussed at the next session

*Book reports

Early in the course, each student reads and makes an informal spoken report to the class on an old book (from an antiquarian collection and/or an online book).

What math was presented in your book?

Who was the intended audience?

What was the purpose of teaching math?

A book report is a student's first major assignment. It is presented orally and rather informally to the class, and it can be lengthy (up to 30 minutes).

(We will briefly see a book report today)

*Final Project

Find something in your book that is interesting to you and **formulate a research question** which you address in your project.

Present it in a 15-min talk at the end of the semester.

And with continued work, submit it to a conference or make it into an article to submit for publication.

Final projects are presented at a mini-conference at the end of the semester, open to the university community. They are 15 to 20 min in length. Students who do not present at the mini-conference do not receive a grade of A in the course.

(We will briefly look at three final projects today)

◆Works of four students from the most recent class (2017)

1. A book report

Elementary Arithmetic in Cherokee and English,
Designed for Beginners
by John B. Jones
1870

A Book Report by Robin Hastings

ᎣᏏᏲ (Osiyo)
(hello in Cherokee)

ELEMENTARY ARITHMETIC,

IN

CHEROKEE AND ENGLISH,

DESIGNED FOR BEGINNERS.

BY JOHN B. JONES.

PREPARED BY AUTHORITY OF THE CHEROKEE NATIONAL COUNCIL.

CHEROKEE NATIONAL PRESS:
TAHLEQUAH, CHEROKEE NATION.

ᏗᏎᏍᏗ ᏗᎴᏅᏙᏗ ᏗᏕᏠᏆᏍᏗ,

ᎢᏧᎳ

ᏣᎳᎩ ᎠᎴ ᎩᎵᏏ ᎤᏂᏬᏂᎯᏍᏗ,

ᎠᎾᎴᏂᏍᎩ ᏧᏅᏙᏗ.

ᏣᏂ ᏚᏬᎯᏳᏍᎩ ᎤᏬᏪᎳᏅᎯ.

ᏣᎳᎩ ᏗᏂᎳᏫᎩ ᎤᏂᏁᏨᎯ.

ᏣᎳᎩ ᎠᏰᎵ ᎤᏂᏯ ᏗᏚᎴᏴᏙᏗᏱ.
ᏗᏚᎴᏴᏔᏅᎯ.

1870.

P R E F A C E

It has long been deplored that that portion of the children of this Nation, who do not speak English, have been compelled to lose entirely the benefits of our Public Schools,...

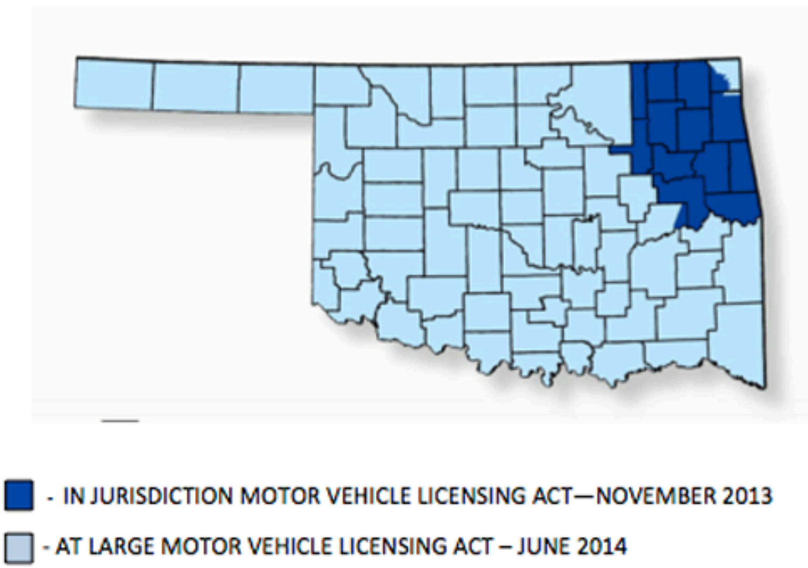
...As one step toward remedying this evil, the National Council, in November, 1869, appointed...a Committee to select an Arithmetic, Geography, and History, to be translated, and published in both the Cherokee and English languages. The Committee has found no Arithmetic suitable to be wholly translated. The work of translation having been entrusted to my hands, I have consulted various authors, and have prepared much of what follows especially for this volume....

May [this little work] prove, to that portion of the people who speak the Cherokee language only, the key to unlock the science of Arithmetic.

JOHN B. JONES

Mr. Hastings’ report focuses on Sequoyah (c. 1770-1843), who invented the Cherokee alphabet (a syllabary of 86 characters), the Cherokee people and their history.
He describes the Trail of tears, the forced relocation of Native American peoples from their ancestral homelands in the Southeast USA to “Indian Territory”, land designated for their use in what is now Oklahoma.

The present Cherokee Nation in NE corner of Oklahoma



What makes this book interesting?

- It contains rules and examples in Cherokee, each followed by translation into English.

- The definitions and rules are clearly written.

- The examples given mostly derive from white culture, but could have been appropriate to the Cherokee, who by 1870 had assimilated many aspects of white American culture.

Addition

ADDITION.

§ 15. 1. If you have 2 cents, and find 3 cents, how many cents will you have? Ans. 5.

2. I spent 12 cents for a slate, and 5 cents for a copy-book; how many cents did I spend? Ans. 17 cents.

3. John gave 6 cents for an orange, 7 cents for a lead pencil, and 9 cents for a ball; how many cents did they all cost? Ans. 22 cents.

§ 16. The process of uniting two or more numbers into one number is called Addition.

The number obtained by the addition is called the sum.

OF THE SIGNS.

§ 17. The sign +, called *plus*, means added. When it stands between two numbers, it shows that they are to be added. Thus: 4 + 2 means that 4 and 2 are to be added together.

The sign = is called the sign of equality, and denotes that the quantities between which it stands equal each other.

The expression: 4 + 2 = 6, means that the sum of 4 and 2 is 6. Read 4 and 2 are six.

ADDITION TABLE.

2	and	1	are	3	3	and	1	are	4	4	and	1	are	5
2	"	2	"	4	3	"	2	"	5	4	"	2	"	6
2	"	3	"	5	3	"	3	"	6	4	"	3	"	7
2	"	4	"	6	3	"	4	"	7	4	"	4	"	8
2	"	5	"	7	3	"	5	"	8	4	"	5	"	9
2	"	6	"	8	3	"	6	"	9	4	"	6	"	10
2	"	7	"	9	3	"	7	"	10	4	"	7	"	11
2	"	8	"	10	3	"	8	"	11	4	"	8	"	12
2	"	9	"	11	3	"	9	"	12	4	"	9	"	13
2	"	10	"	12	3	"	10	"	13	4	"	10	"	14
2	"	11	"	13	3	"	11	"	14	4	"	11	"	15
2	"	12	"	14	3	"	12	"	15	4	"	12	"	16

Addition in Cherokee

ᏕᏟᏬᏗᎽ.

§ 15. 1. ᎢᏳᏃ 2 ᎢᏯᏓᏅᏖᏗ ᎽᏗᏳᏌ, 3Ꮓ ᎢᏯᏓᏅᏖᏗ ᎽᏗᏳᏣᏰᏠ, ᎯᎳᎪ ᎢᏯᏓᏅᏖᏗ ᎽᏗᏳᏌ?

ᏕᏫᎯᏝᏴᏍᎡᎢ 5 ᎢᏯᏓᏅᏖᏗ.

2. ᏇᏅᏀᏕᏄ ᏗᎪᏨᏳᏗ ᎢᏯᏓᏅᏖᏗ ᎠᎢᏗᏴᏠᎩ, Ꮓ ᎢᏯᏓᏅᏖᏗ Ꭺ-ᏨᏝ ᏗᎪᏨᏳᏗ; ᎯᎳᎪ ᎢᏯᏓᏅᏖᏗ ᏓᎢᏗᏴᎮᎢ? ᏕᏫᎯᏝᏴᏍᎡ 1, ᎢᏯᏓᏅ-ᏖᏗ.

3. ᏣᏂ 6 ᎢᏯᏓᏅᏖᏗ ᎤᏗᏴᏠᎩ ᏧᏥᏂ, Ꮓ ᎢᏯᏓᏅᏖᏗ ᎤᏗᏴᏠᎩ ᏕᏂ ᏗᎪᏨᏳᏗ, Ꮓ ᎢᏯᏓᏅᏖᏗ ᎤᏗᏴᏠᎩ ᎠᏔᏍᏕᏳᏗ; ᎯᎳᎪ ᎢᏯᏓᏅᏖᏗ Ꮝ-ᏝᎬᏩᏔᏁ ᏂᏕᎠ? ᏕᏫᎯᏝᏴᏍᎡ 2 ᎢᏯᏓᏅᏖᏗ.

§ 16. ᏔᏝ ᏗᏎᏍᏗ, ᎠᎴ ᎤᏗᏗᏢ ᎢᏕᎢ ᎽᎾᎩ, ᎽᎠᏜᏍᏗᏌ, ᏕᏟᏬᏗᎽ ᏚᏙᎥ. ᎾᏍᎩᏃ ᏕᏟᏥᏅᎯ ᎢᏕᎢ ᎾᏂᏕᏝᏍᏔᏂᎯᏠᎢ ᏂᏕᎡᎢ, ᏚᏙᎥ.

ᏗᏎᏍᎩ ᏓᏍᏛᏛᎢ.

§ 17. ᎯᎠ ᏓᏍᏓᏅᏛᎢ +, Plus, ᏚᏙᎥ, ᎾᏍᎾᏢᏗ ᏕᎤᏕ.

ᏔᏝᏃ ᏗᏎᏍᏗ ᏕᎪᏨᏄ ᎠᏰᏝ ᏂᏕᏙᎪ ᎾᏍᎩ ᏕᏟᏬᏗ ᏥᎡ ᏬᏃᏠᏍᎪᎢ; ᎾᏍᎩ ᎯᎠ 4 +, ᏬᏃᏠᏍᏕ ᎠᎴ ᏗᏕᏟᏬᏗ ᏥᎡᎢ.

ᎯᎠ ᏓᏍᏓᏅᏛᎢ = ᎢᏧᏔᎲ ᏥᎡ ᏬᏃᏠᏍᎩ ᏚᏙᎥ, ᎾᏍᎩᏃ ᏬᏃᏠᏍᎪᎢ ᎾᏍᎩ ᏗᏎᏍᏗ ᏕᎪᏨᏄ ᎾᏕ ᎠᏰᏝ ᏥᎡ ᏕᏙᎡᎢ, ᎢᏧᏔᎲ ᎢᏗᏕᏕᎢᏠ ᏥᎡᎢ.

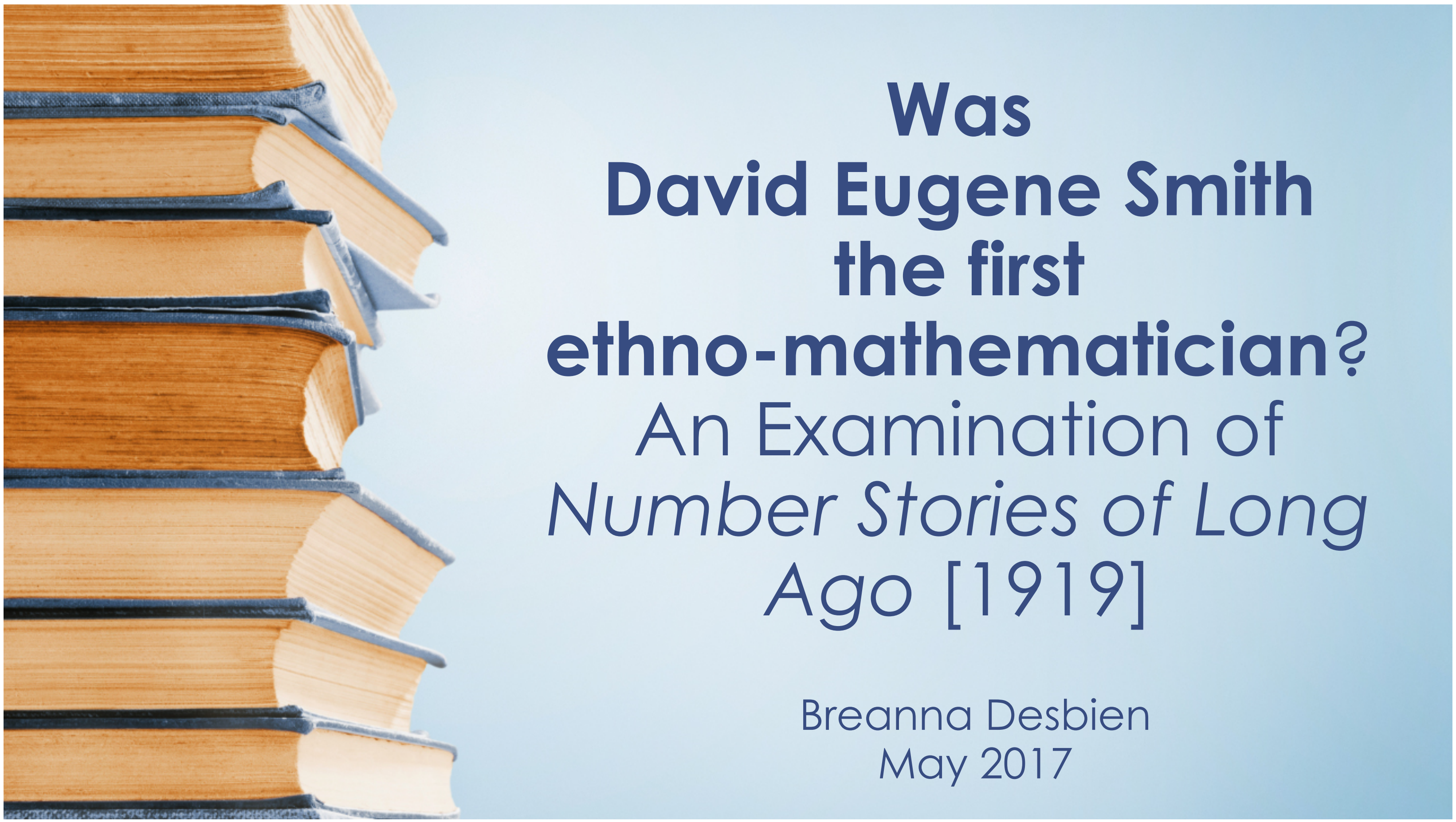
ᎯᎠ ᏥᏂᎬᏅ 4 + 2 = , ᎯᎠ ᏄᏍᏗ ᏕᎤᏕ, ᎾᏍᎩ 4, 2Ꮓ ᏗᏕᏟᏥᏅ 6 ᏂᏕᏕ. ᎠᎪᏝᏰᏗᏃ ᎯᎠ ᏄᏍᏗ = Ꮓ ᏂᏕᏝᏍᏗᎲ.

ᏕᏟᏬᏗ ᎤᎬᏩᏝ ᏕᏕᏅᏃᏠᎢ.

2	ᎠᎴ	1	3	ᎢᎩ	3	ᎠᎴ	1	4	ᎢᎩ	4	ᎠᎴ	1	5	ᎢᎩ
2	‘‘	2	4	‘‘	3	‘‘	2	5	‘‘	4	‘‘	2	6	‘‘
2	‘‘	3	5	‘‘	3	‘‘	3	6	‘‘	4	‘‘	3	7	‘‘
2	‘‘	4	6	‘‘	3	‘‘	4	7	‘‘	4	‘‘	4	8	‘‘
2	‘‘	5	7	‘‘	3	‘‘	5	8	‘‘	4	‘‘	5	9	‘‘
2	‘‘	6	8	‘‘	3	‘‘	6	9	‘‘	4	‘‘	6	10	‘‘
2	‘‘	7	9	‘‘	3	‘‘	7	10	‘‘	4	‘‘	7	11	‘‘
2	‘‘	8	10	‘‘	3	‘‘	8	11	‘‘	4	‘‘	8	12	‘‘
2	‘‘	9	11	‘‘	3	‘‘	9	12	‘‘	4	‘‘	9	13	‘‘
2	‘‘	10	12	‘‘	3	‘‘	10	13	‘‘	4	‘‘	10	14	‘‘
2	‘‘	11	13	‘‘	3	‘‘	11	14	‘‘	4	‘‘	11	15	‘‘
2	‘‘	12	14	‘‘	3	‘‘	12	15	‘‘	4	‘‘	12	16	‘‘

ᏙᎾᏓᎬᎰᎢ (Donadagvhoi)
(Goodbye)

2. A final project:
Was David Eugene Smith the first Ethno-mathematician? An Examination of *Number Stories of Long Ago* (1915) (Breanna Desbien)



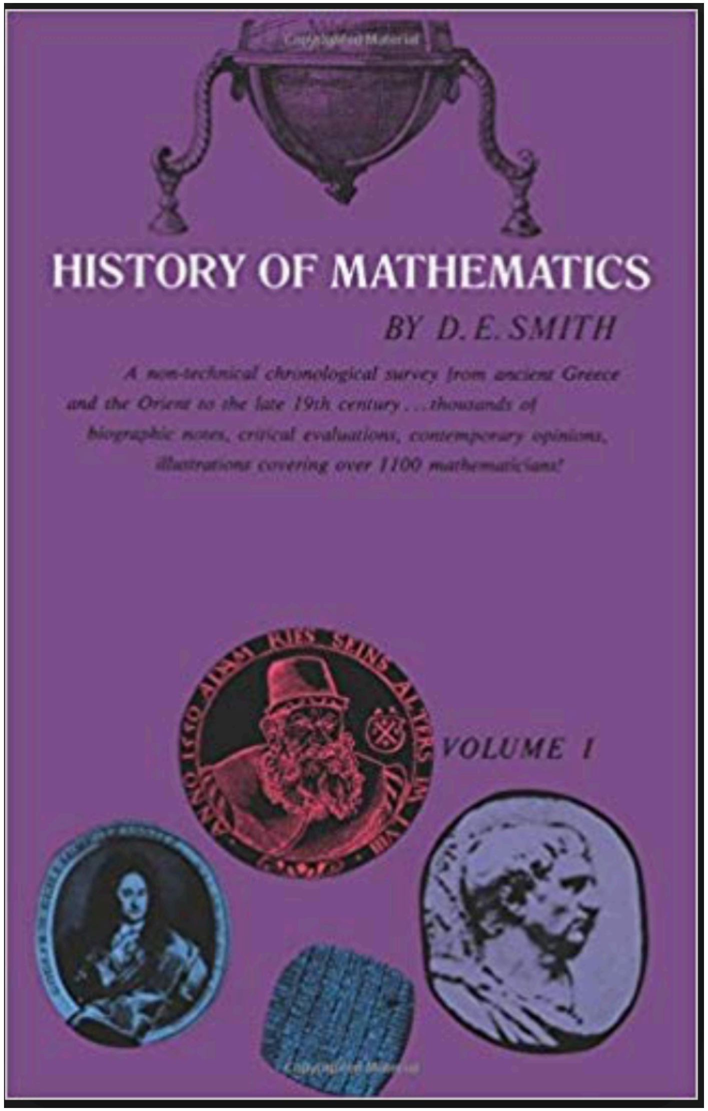
Biography of D.E. Smith

January 1860-July 1944

Obtained **Ph.D. in 1887** from Syracuse University.

He was the **Mathematical Association of America President** in 1920.

He **wrote many mathematics textbooks** and pieces on the **history of mathematics**.


HISTORY OF MATHEMATICS
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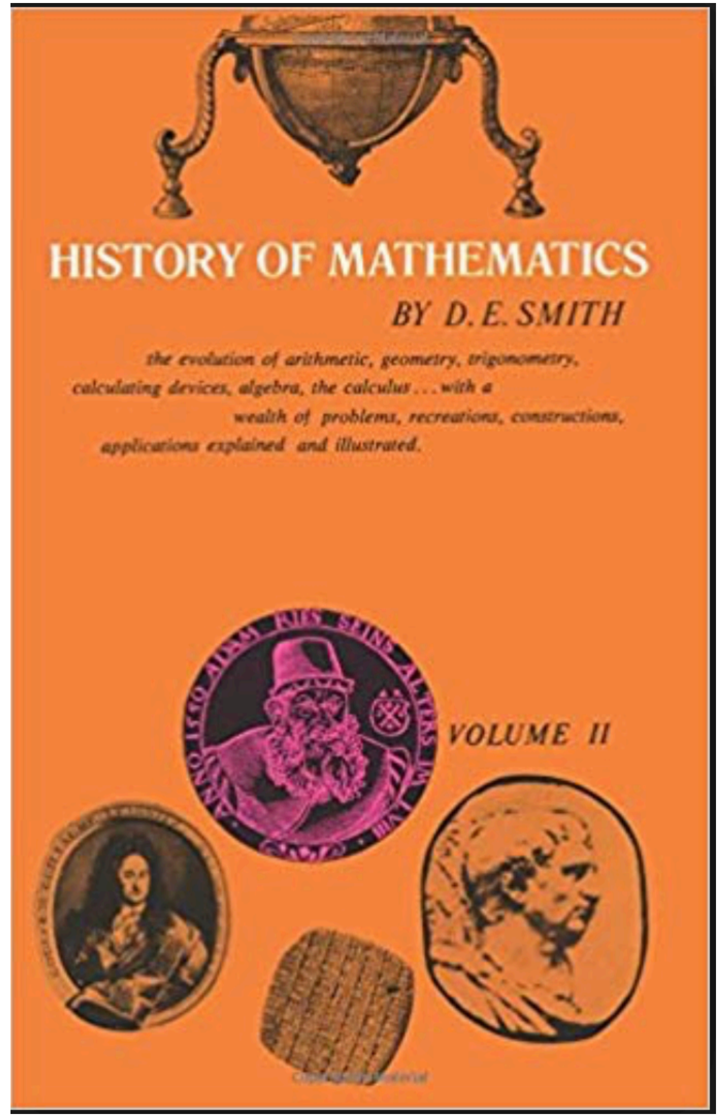
VOLUME I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NUMBER STORIES OF LONG AGO

BY

DAVID EUGENE SMITH

GINN AND COMPANY
BOSTON · NEW YORK · CHICAGO · LONDON
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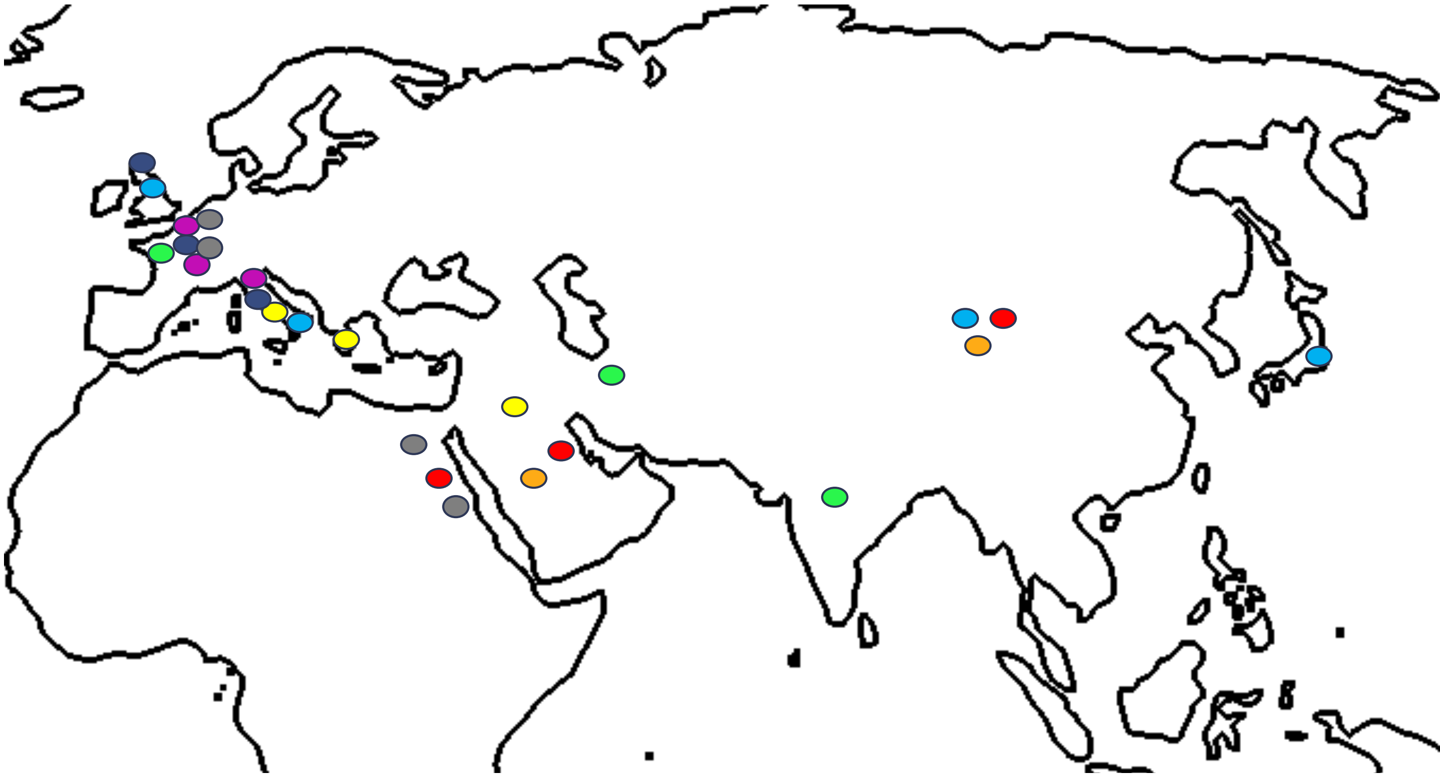
CONTENTS

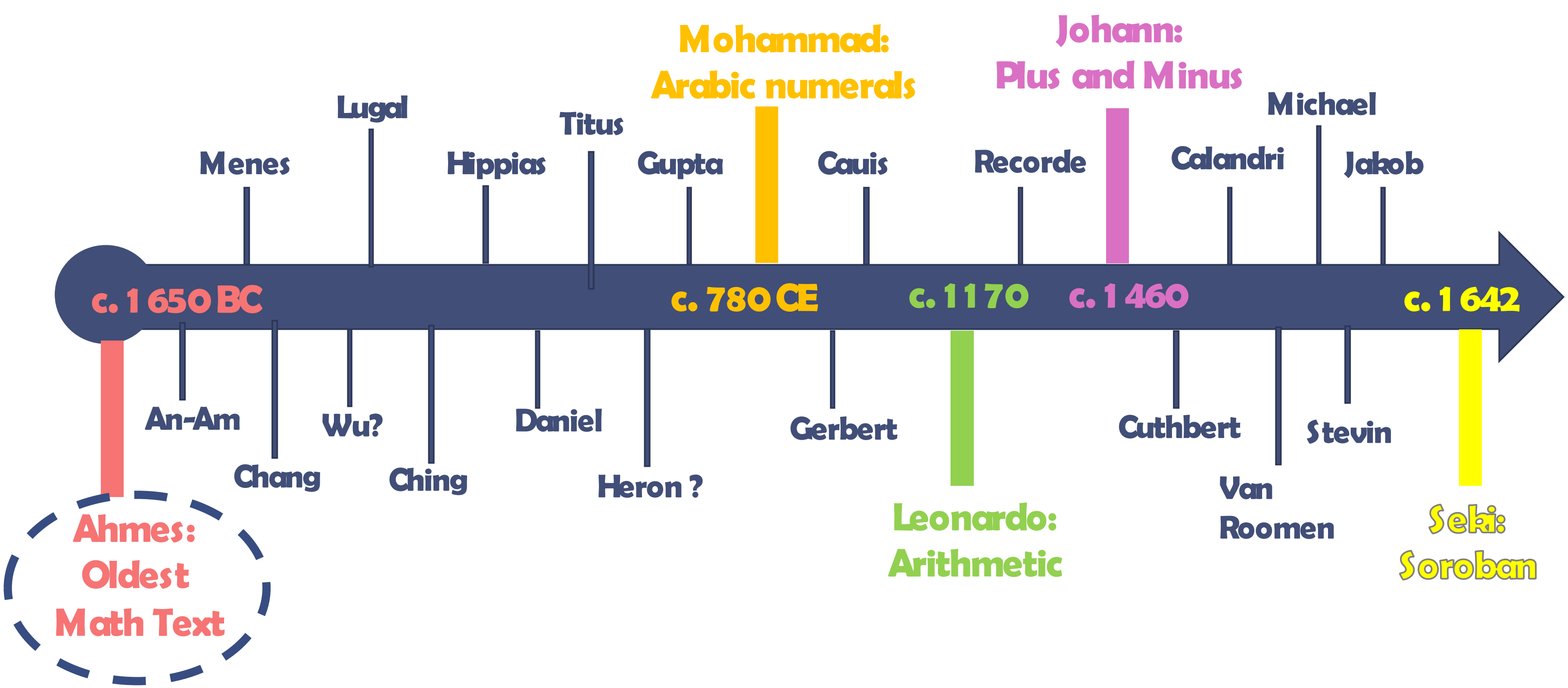
Number Stories of Long Ago:
Map of Characters

<u>Key</u>

- Ch. 1 Counting
- Ch. 2 Writing
- Ch. 3 Writing
- Ch. 4 Writing
- Ch. 5 Adding
- Ch. 6 Multiply
- Ch. 7 Divide
- Ch. 8 Fractions

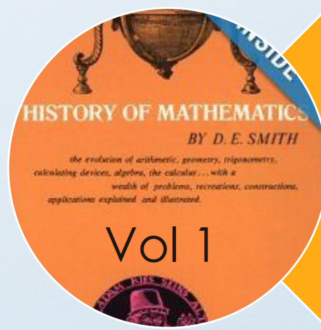


Character **Timeline**



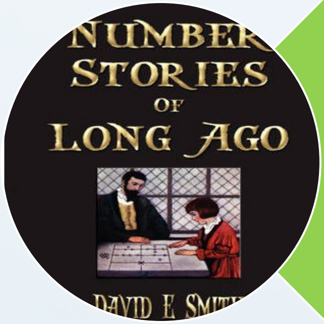
Al Khwarizmi c. 780 CE

"He believed that [Hindu numerals] were better than the ones the Arabs used, and so he wrote a book about them." pg. 39

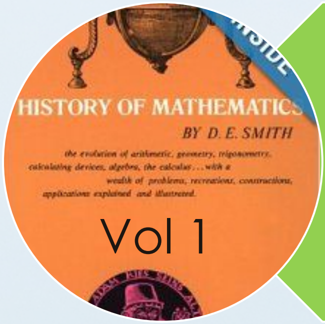


"He is best known for having written the first work bearing the name 'algebra'." pg. 170

Leonardo Fibonacci c.1170



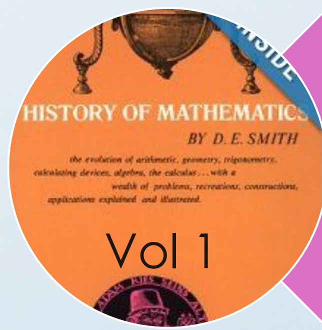
"When he became a man he described [Hindu numerals] in a book which he wrote, and this assisted in making them known." pg. 40



"The first great mathematician of the 13th century, and indeed the greatest and most productive mathematician of all the Middle Ages…" pg. 214

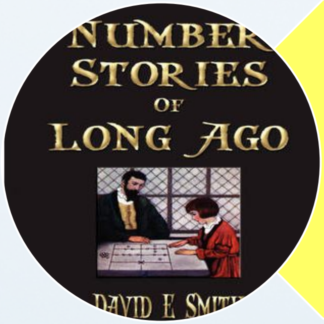
Johann Widman c. 1460

"When Johann became a man he wrote an arithmetic, and in this he used the signs for plus (+) and minus (—)." pg. 69

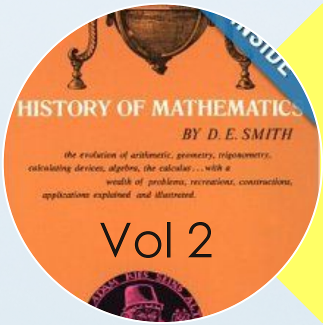


"He wrote the first important German textbook on commercial arithmetic, and in this appear for the very first time the signs + and -…" pg. 258

Seki Kōwa c. 1642



"Seki grew up to become the greatest mathematician of Japan…" pg. 55



"[He] is known to have written a work called the Kai Fukudai no Hō in 1683." pg. 477

Seki Takakazu (Seki Kōwa)

Seki Takakazu (Seki Kōwa)

Conclusion

Is D. E. Smith an ethno-mathematician?
I think so.

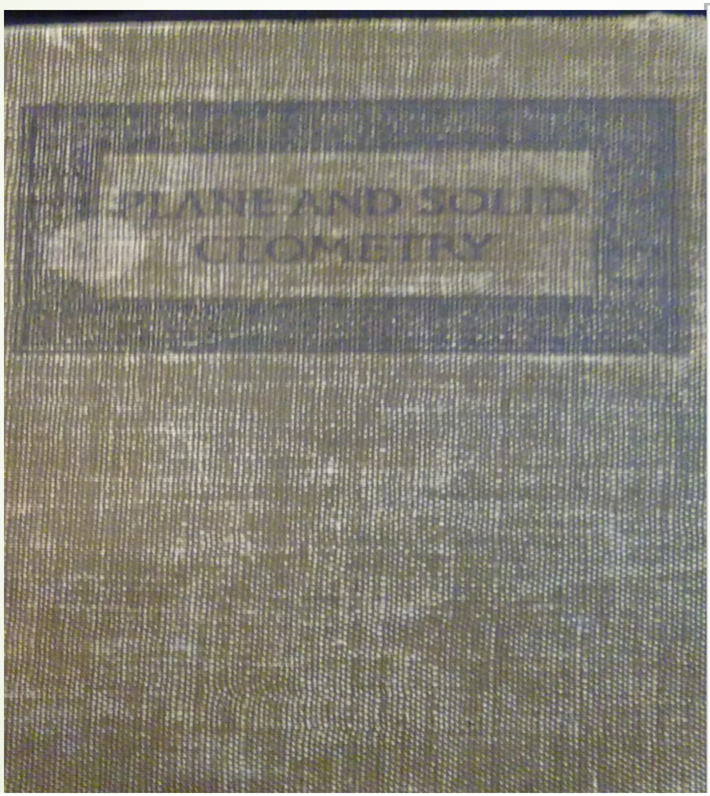
Is D.E. Smith the first ethno-mathematician?
I believe so, or at least the first ethno-mathematician whose work is accessible to children.

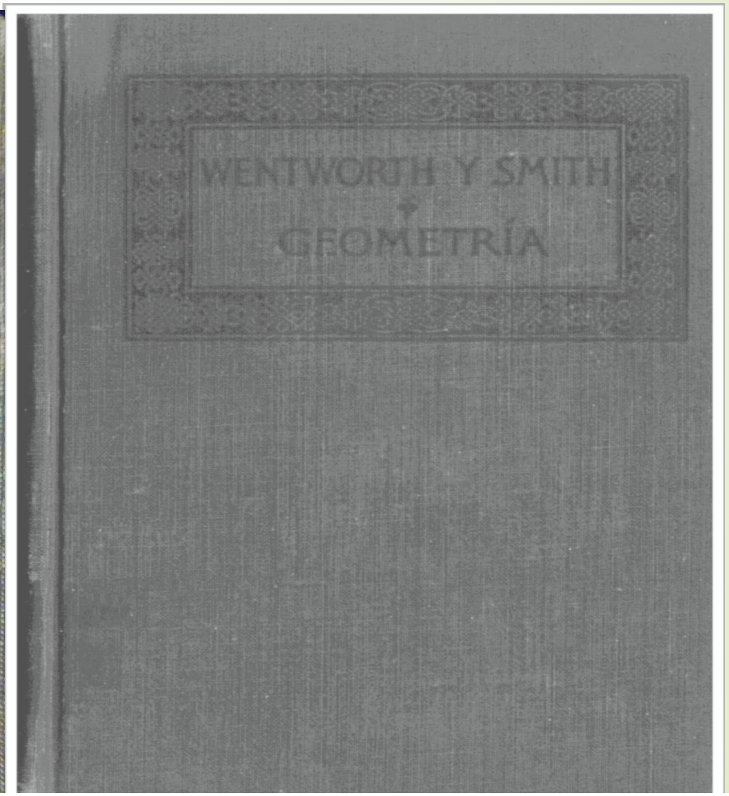
3. A final project: Comparing *Geometria Plana y del Espacio* (1915) and *Plane and Solid Geometry* (1913), both by G. A. Wentworth and David Eugene Smith (Jose Terrazas-Reyes)

Research Question

- Was the quality of the book and the level of difficulty preserved in the Spanish version after translating from English to Spanish?

Covers





Title Page

WENTWORTH-SMITH MATHEMATICAL SERIES

PLANE AND SOLID GEOMETRY

BY

GEORGE WENTWORTH

AND

DAVID EUGENE SMITH

GINN AND COMPANY

BOSTON · NEW YORK · CHICAGO · LONDON
ATLANTA · DALLAS · COLUMBUS · SAN FRANCISCO

SERIE MATEMÁTICA WENTWORTH Y SMITH

GEOMETRÍA
PLANA Y DEL ESPACIO

POR

JORGE WENTWORTH

Y

DAVID EUGENIO SMITH

GINN Y COMPAÑÍA

BOSTON · NUEVA YORK · CHICAGO · LONDRES

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PLANE GEOMETRY

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TABLA GENERAL DE MATERIAS

[El índice alfabético se halla al fin de la obra]

GEOMETRÍA PLANA

v

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SOLID GEOMETRY

GEOMETRÍA DEL ESPACIO

Translation Note

El traductor ha adoptado algunos términos y expresiones, como *perpendicular bisectriz* y *bisectar* (la Academia trae *bisecar*), que, si bien inusitados o poco usados en castellano, son de indisputable utilidad y no pueden tacharse de incorrectos. También ha seguido el sistema inglés y norteamericano de escribir las abreviaturas referentes a decimales después del número completo y no después de los enteros: 4,36 km., y no 4^{km}, 36.

The translator has adopted some terms and expressions, such as perpendicular bisector and bisectrix (the Academy brings bisect), which, although unusual or little used in Castillian, are indisputably useful and can not be labeled as incorrect. He has also followed the English and North American system of writing the abbreviations referring to decimals after the complete number and not after the integers: 4.36 km., and not 4 km, 36.

Common Symbols and Abbreviations

SYMBOLS AND ABBREVIATIONS

=	equals, equal, equal to, is equal to, or is equivalent to.	Adj.	adjacent.
>	is greater than.	Alt.	alternate.
<	is less than.	Ax.	axiom.
∥	parallel.	Const.	construction.
⊥	perpendicular.	Cor.	corollary.
∠	angle.	Def.	definition.
△	triangle.	Ex.	exercise.
▱	parallelogram.	Ext.	exterior.
□	rectangle.	Fig.	figure.
⊙	circle.	Hyp.	hypothesis.
st.	straight.	Iden.	identity.
rt.	right.	Int.	interior.
∵	since.	Post.	postulate.
∴	therefore.	Prob.	problem.
		Prop.	proposition.
		Sup.	supplementary.

These symbols take the plural form when necessary, as in the case of ∥s, ∠s, △s, ⊙s.

The symbols +, −, ×, ÷ are used as in algebra.

There is no generally accepted symbol for "is congruent to," and the words are used in this book. Some teachers use ≏ or ≅, and some use ≡, but the sign of equality is more commonly employed, the context telling whether equality, equivalence, or congruence is to be understood.

Q. E. D. is an abbreviation that has long been used in geometry for the Latin words *quod erat demonstrandum*, "which was to be proved."

Q. E. F. stands for *quod erat faciendum*, "which was to be done."

ABREVIATURAS Y SÍMBOLOS

Circunf.	circunferencia.	>	mayor que.
Constr.	construcción.	<	menor que.
Fig.	figura.	∥	paralelo, paralela.
Hipót.	hipótesis.	⊥	perpendicular.
Ident.	identidad.	∠	ángulo, el ángulo.
L.C.D.D.	lo cual debíamos demostrar.	△	triángulo, el triángulo
N.°	número.	▱	paralelogramo.
Prop.	proposición.	□	rectángulo.
rt.	recto, rectos.	⊙	círculo.
		∴	luego, de donde.

El plural de una palabra representada por un símbolo se indica poniendo una *s* al símbolo: ∠s, *ángulos, los ángulos*; ⊥s, *perpendiculares*; ∥s, *paralelos, paralelas*; △s, *triángulos, los triángulos*.

General Comparison Slides

PLANE GEOMETRY

INTRODUCTION

1. The Nature of Arithmetic. In arithmetic we study computation, the working with numbers. We may have a formula expressed in algebraic symbols, such as $a = bh$, where a may stand for the area of a rectangle, and b and h respectively for the number of units of length in the base and height; but the actual computation involved in applying such formula to a particular case is part of arithmetic.

2. The Nature of Algebra. In algebra we generalize the arithmetic, and instead of saying that the area of a rectangle with base 4 in. and height 2 in. is 4×2 sq. in., we express a general law by saying that $a = bh$. In arithmetic we may have an equality, like $2 \times 16 + 17 = 49$, but in algebra we make much use of equations, like $2x + 17 = 49$. Algebra, therefore, is a generalized arithmetic.

3. The Nature of Geometry. We are now about to begin another branch of mathematics, one not chiefly relating to numbers although it uses numbers, and not primarily devoted to equations although using them, but one that is concerned principally with the study of forms, such as triangles, parallelograms, and circles. Many facts that are stated in arithmetic and algebra are proved in geometry. For example, in geometry it is proved that the square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides, and that the circumference of a circle equals 3.1416 times the diameter.

GEOMETRÍA PLANA

INTRODUCCIÓN

1. Carácter de la aritmética. En la aritmética se estudian los cálculos numéricos. Hácese a veces uso de las fórmulas; mas el objeto de la aritmética no es tanto el establecerlas como el enseñar a ejecutar las operaciones necesarias para aplicarlas.

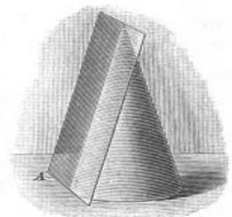
2. Carácter del álgebra. En el álgebra se generalizan las cuestiones de aritmética, y por lo común se expresan las reglas y teoremas por medio de fórmulas. Así, en vez de decir que el área de un rectángulo es igual al producto de la base por la altura, se establece la fórmula $A = bh$. Aun cuando la aritmética se vale algunas veces de las ecuaciones, no lo hace tan a menudo como el álgebra, que resuelve casi todo problema por medio de ellas. En resumen, el álgebra es una extensión y generalización de la aritmética.

3. Carácter de la geometría. Vamos ahora a entrar en un ramo de las matemáticas que difiere radicalmente de la aritmética y el álgebra; ramo que, si bien hace uso frecuente de cálculos numéricos, ecuaciones y fórmulas, tiene por objeto principal el estudio de las *formas* o *figuras*, tales como rectángulos, triángulos y círculos, de que la aritmética y el álgebra no dan más que ideas muy generales, y cuyas propiedades se enuncian en estas ciencias, pero no se demuestran. Toca a la geometría dar demostraciones de tales propiedades: así, ella demuestra rigorosamente que el cuadrado construído sobre la hipotenusa de un triángulo rectángulo es igual a la suma de los cuadrados construídos sobre los catetos.

General Comparison Slides

603. Tangent Plane. A plane which contains an element of a cone, but does not cut the surface, is called a *tangent plane* to the cone.

604. Construction of Tangent Planes. It is evident that:

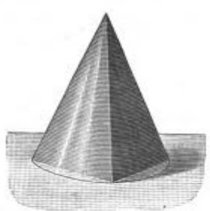


A plane passing through a tangent to the base of a circular cone and the element drawn through the point of contact is tangent to the cone.

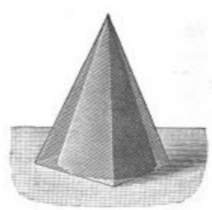
If a plane is tangent to a circular cone its intersection with the plane of the base is tangent to the base.

605. Inscribed Pyramid. A pyramid whose lateral edges are elements of a cone and whose base is inscribed in the base of the cone is called an *inscribed pyramid*.

In this case the cone is said to be *circumscribed* about the pyramid.



Inscribed Pyramid



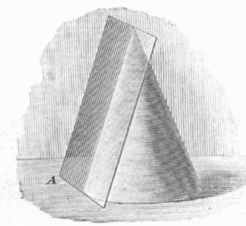
Circumscribed Pyramid

606. Circumscribed Pyramid. A pyramid whose lateral faces are tangent to the lateral surface of a cone and whose base is circumscribed about the base of the cone is called a *circumscribed pyramid*.

In this case the cone is said to be *inscribed* in the pyramid.

603. Plano tangente. Llámase *plano tangente* a un cono todo plano que contiene una generatriz pero no corta la superficie del cono.

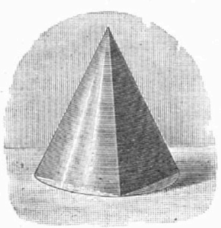
604. Construcción de planos tangentes. Es evidente que en un cono circular cualquiera,



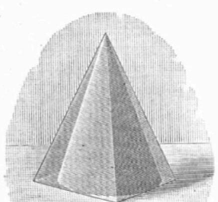
Todo plano que contiene una tangente a la base y la generatriz que pasa por el punto de contacto es tangente al cono;

Todo plano tangente corta el de la base según una tangente a la base.

605. Pirámide inscrita en un cono. Dícese que una pirámide está *inscrita* en un cono cuando sus aristas laterales son generatrices del cono y su base está inscrita en la del cono. Dícese también que el cono está entonces *circunscrito* a la pirámide.



Pirámide inscrita



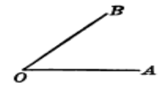
Pirámide circunscrita

606. Pirámide circunscrita a un cono. Una pirámide está *circunscrita* a un cono cuando sus caras son tangentes al cono y su base está circunscrita a la del cono. El cono está entonces *inscrito* en la pirámide.

Terms and Definitions Comparison

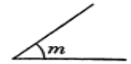
21. Angle. The opening between two straight lines drawn from the same point is called an *angle*.

Strictly speaking, this is a *plane* angle. We shall find later that there are angles made by curve lines and angles made by planes.



The two lines are called the *sides* of the angle, and the point of meeting is called the *vertex*.

An angle may be read by naming the letters designating the sides, the vertex letter being between the others, as the angle AOB. An angle may also be designated by the vertex letter, as the angle O, or by a small letter within, as the angle m. A curve is often drawn to show the particular angle meant, as in angle m.

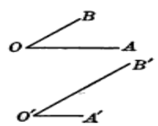


22. Size of Angle. The size of an angle depends upon the amount of turning necessary to bring one side into the position of the other.

One angle is greater than another angle when the amount of turning is greater. Thus in these compasses the first angle is smaller than the second, which is also smaller than the third. The length of the sides has nothing to do with the size of the angle.

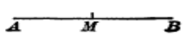
23. Equality of Angles. Two angles that can be placed one upon the other so that their vertices coincide and the sides of one lie along the sides of the other are said to be *equal*.

For example, the angles AOB and $A'O'B'$ (read "A prime, O prime, B prime") are equal. It is well to illustrate this by tracing one on thin paper and placing it upon the other.



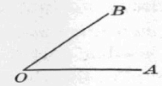
24. Bisector. A point, a line, or a plane that divides a geometric magnitude into two equal parts is called a *bisector* of the magnitude.

For example, M, the mid-point of the line AB, is a bisector of the line.

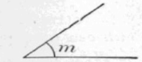


21. Ángulo. Llámase *ángulo* la abertura entre dos rectas que se encuentran.

Las dos rectas que se encuentran se llaman *lados* del ángulo, y el punto en que se encuentran, *vértice* del ángulo.

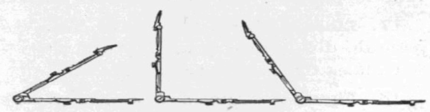


Un ángulo puede nombrarse por tres letras, una escrita en cada uno de los lados, y la otra en el vértice. La del vértice se nombra entre las otras dos: ángulo AOB. También se nombra un ángulo por medio de una letra puesta en el vértice: ángulo O; o entre los lados del ángulo: ángulo m. A veces, para mayor claridad, se traza una curva entre los lados del ángulo, y sobre ella se escribe la letra, como se ve en m.



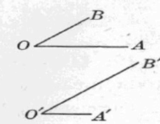
22. Tamaño o magnitud de un ángulo. La magnitud de un ángulo depende únicamente de la magnitud del movimiento necesario para llevar un lado, haciéndolo girar sobre el vértice, a la posición del otro. Este movimiento se llama *rotación*.

Un ángulo es mayor que otro cuando la rotación es mayor en aquél que en éste. Así, para las tres aberturas del compás aquí representado, el primer ángulo es menor que el segundo, y el segundo menor que el tercero. La longitud de los lados no afecta la magnitud de un ángulo.

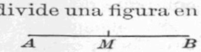


23. Igualdad de los ángulos. Dos ángulos son *iguales* cuando el uno puede colocarse sobre el otro de manera que los vértices coincidan y los lados del uno queden sobre los del otro.

Por ejemplo, los ángulos AOB y $A'O'B'$ son iguales.



24. Bisector, bisectriz. El adjetivo *bisector* (femenino *bisectriz*) se aplica a todo punto, línea o plano que divide una figura en dos partes iguales. En el caso de la recta, la palabra *bisectriz* se emplea como sustantivo, y así se dice la *bisectriz* de un ángulo en vez de la *línea bisectriz*.



En esta figura, M, punto medio de AB, es el punto bisector de AB.

Places Comparison

22. A double tennis court is 78 ft. long and 36 ft. wide. The net is placed 39 ft. from each end and the service lines 18 ft. from each end. Draw the plan, using a scale of $\frac{1}{16}$ in. to a foot, making the right angles as shown in Ex. 1. The accuracy of the construction may be tested by measuring the diagonals, which should be equal.

23. At the entrance to New York harbor is a gun having a range of 12 mi. Draw a line inclosing the range of fire, using a scale of $\frac{1}{16}$ in. to a mile.

24. Two forts are placed on opposite sides of a harbor entrance, 13 mi. apart. Each has a gun having a range of 10 mi. Draw a plan showing the area exposed to the fire of both guns, using a scale of $\frac{1}{16}$ in. to a mile.

25. Two forts, A and B, are placed on opposite sides of a harbor entrance, 16 mi. apart. On an island in the harbor, 12 mi. from A and 11 mi. from B, is a fort C. The fort A has a gun with a range of 12 mi., fort B one with a range of 11 mi., and fort C one with a range of 10 mi. Draw a plan of the entrance to the harbor, showing the area exposed to the fire of each gun.

26. A horse, tied by a rope 25 ft. long at the corner of a lot 50 ft. square, grazes over as much of the lot as possible. The next day he is tied at the next corner, the third day at the third corner, and the fourth day at the fourth corner. Draw a plan showing the area over which he has grazed during the four days, using a scale of $\frac{1}{4}$ in. to 5 ft.

27. A gardener laid out a flower bed on the following plan: He made a triangle ABC, 16 ft. on a side, and then bisected two of the angles. From the point of intersection of the bisectors, P, he drew perpendiculars to the three sides of the triangle, PX, PY, and PZ. Then he drew a circle with P as a center and PX as a radius, and found that it just fitted in the triangle. Draw the plan, using a scale of $\frac{1}{4}$ in. to a foot.

22. Uno de los pisos de un edificio es un gran cuarto de 25 m. de largo por 15 m. de ancho, dividido al través en tres compartimientos por tabiques, uno en el centro, y cada uno de los otros a 6 m. del extremo respectivo. Dibújese el plano según escala de 4 mm. por metro, representando los tabiques por rectas. Empléese para la construcción de los ángulos rectos el método del ejercicio n.º 1.

23. A la entrada de una bahía hay un cañón cuyo alcance es de 20 km. Trácese una línea que encierre todos los puntos comprendidos dentro del alcance del cañón, empleando una escala de 1,5 mm. por kilómetro.

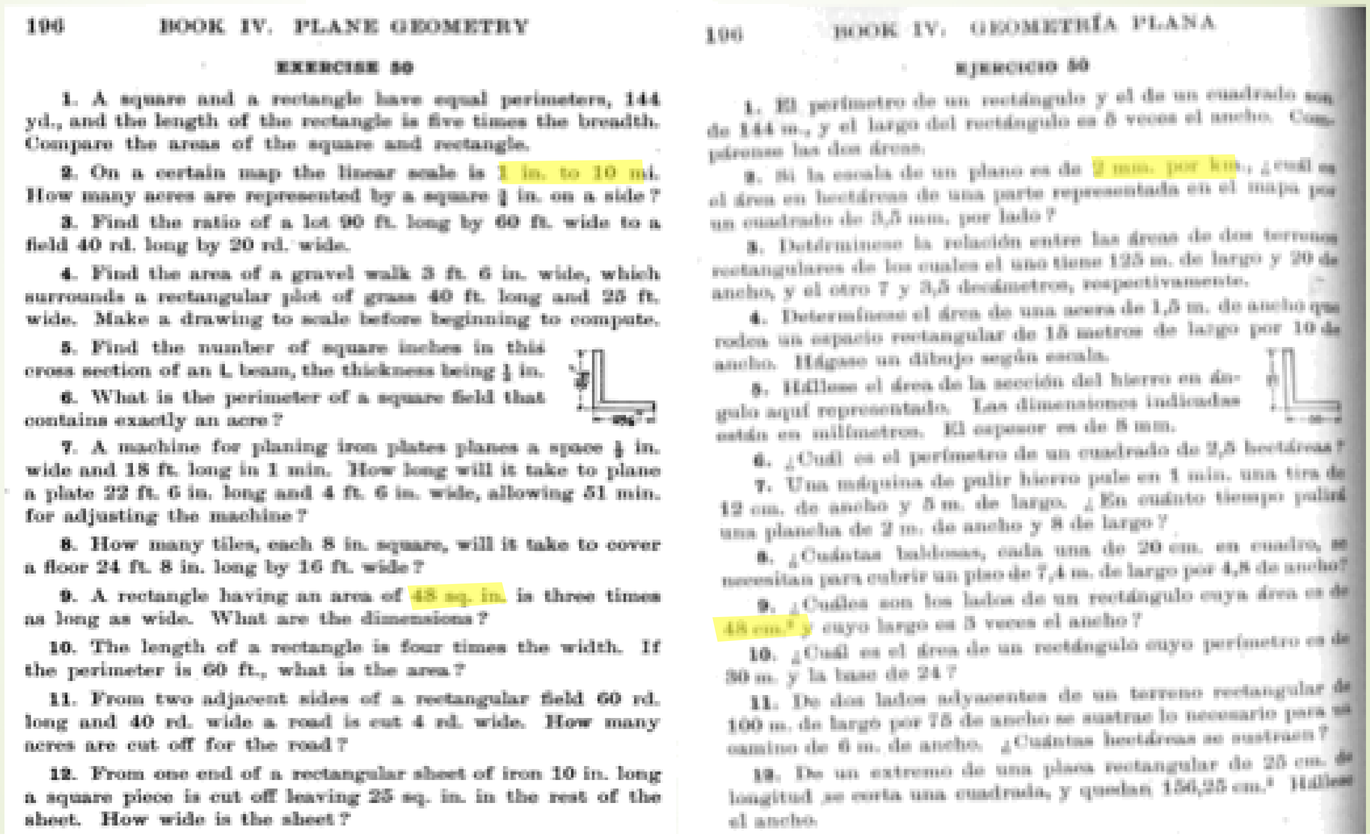
24. Dos fuertes distan 26 km. el uno del otro, y tienen cañones de 16 km. de alcance. Dibújese, en escala de 1,5 mm. por kilómetro, el campo común al alcance de los dos cañones.

25. A la entrada de una bahía, en lados opuestos, hay dos fuertes A y B, que distan 24 km. entre sí. A 18 km. de A y 16,5 km. de B, hay en la bahía una isla con un fuerte C. Los cañones de A, B, C tienen respectivamente alcances de 18, 16,5 y 15 km. Dibújese, según escala conveniente, un plano que represente los tres fuertes y el campo de acción de cada cañón.

26. Un corral cuadrado de 16 m. por lado tiene cuatro postes en las esquinas. Un caballo se ata durante un día a uno de los postes con 8 m. de cuerda, durante otro día a otro de los postes, y así de los otros dos. Dibújese, según escala de 2,5 mm. por metro, un plano que indique la superficie en que el caballo puede pacer durante los cuatro días.

27. Un jardinero proyectó parte de un jardín así: Primero construyó un triángulo ABC, con lados de 4,8 m., y luego bisectó dos de los ángulos. Del punto P de intersección de las bisectrices trazó las perpendiculares PX, PY, PZ a los tres lados del triángulo. Luego trazó un círculo de P como centro, y con radio PX, el cual resultó perfectamente ajustado dentro del triángulo. Hágase esta construcción en escala de 1 : 50

Units Comparison



EXERCISE 50

1. A square and a rectangle have equal perimeters, 144 yd., and the length of the rectangle is five times the breadth. Compare the areas of the square and rectangle.

2. On a certain map the linear scale is 1 in. to 10 mi. How many acres are represented by a square $\frac{3}{4}$ in. on a side?

3. Find the ratio of a lot 90 ft. long by 60 ft. wide to a field 40 rd. long by 20 rd. wide.

4. Find the area of a gravel walk 3 ft. 6 in. wide, which surrounds a rectangular plot of grass 40 ft. long and 25 ft. wide. Make a drawing to scale before beginning to compute.

5. Find the number of square inches in this cross section of an L beam, the thickness being $\frac{1}{2}$ in.

6. What is the perimeter of a square field that contains exactly an acre?

7. A machine for planing iron plates planes a space $\frac{1}{4}$ in. wide and 18 ft. long in 1 min. How long will it take to plane a plate 22 ft. 6 in. long and 4 ft. 6 in. wide, allowing 51 min. for adjusting the machine?

8. How many tiles, each 8 in. square, will it take to cover a floor 24 ft. 8 in. long by 16 ft. wide?

9. A rectangle having an area of 48 sq. in. is three times as long as wide. What are the dimensions?

10. The length of a rectangle is four times the width. If the perimeter is 60 ft., what is the area?

11. From two adjacent sides of a rectangular field 60 rd. long and 40 rd. wide a road is cut 4 rd. wide. How many acres are cut off for the road?

12. From one end of a rectangular sheet of iron 10 in. long a square piece is cut off leaving 25 sq. in. in the rest of the sheet. How wide is the sheet?

EJERCICIO 50

1. El perímetro de un rectángulo y el de un cuadrado son de 144 m., y el largo del rectángulo es 5 veces el ancho. Compárense las dos áreas.

2. Si la escala de un plano es de 2 mm. por km., ¿cuál es el área en hectáreas de una parte representada en el mapa por un cuadrado de 3,5 mm. por lado?

3. Determínese la relación entre las áreas de dos terrenos rectangulares de los cuales el uno tiene 125 m. de largo y 20 de ancho, y el otro 7 y 3,5 decámetros, respectivamente.

4. Determínese el área de una acera de 1,5 m. de ancho que rodea un espacio rectangular de 15 metros de largo por 10 de ancho. Hágase un dibujo según escala.

5. Hállese el área de la sección del hierro en ángulo aquí representado. Las dimensiones indicadas están en milímetros. El espesor es de 8 mm.

6. ¿Cuál es el perímetro de un cuadrado de 2,5 hectáreas?

7. Una máquina de pulir hierro pule en 1 min. una tira de 12 cm. de ancho y 5 m. de largo. ¿En cuánto tiempo pulirá una plancha de 2 m. de ancho y 8 de largo?

8. ¿Cuántas baldosas, cada una de 20 cm. en cuadro, se necesitan para cubrir un piso de 7,4 m. de largo por 4,8 de ancho?

9. ¿Cuáles son los lados de un rectángulo cuya área es de 48 cm.^{2} y cuyo largo es 3 veces el ancho?

10. ¿Cuál es el área de un rectángulo cuyo perímetro es de 80 m. y la base de 24?

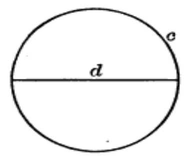
11. De dos lados adyacentes de un terreno rectangular de 100 m. de largo por 75 de ancho se sustrae lo necesario para un camino de 6 m. de ancho. ¿Cuántas hectáreas se sustraen?

12. De un extremo de una placa rectangular de 25 cm. de longitud se corta una cuadrada, y quedan 156,25 cm.^{2} Hállese el ancho.

Advanced Geometry

PROPOSITION XV. PROBLEM

404. *To find the numerical value of the ratio of the circumference of a circle to its diameter.*



Given a circle of circumference c and diameter d.

Required to find the numerical value of $\frac{c}{d}$ *or* π.

Solution. By § 385, $2\pi r = c$. $\therefore \pi = \frac{1}{2}c$ when $r = 1$.

Let s_6 (read "s sub six") be the length of a side of a regular polygon of 6 sides, s_{12} of 12 sides, and so on.

If $r = 1$, by § 394, $s_6 = 1$, and by § 403 we have

Form of Computation	Length of Side	Length of Perimeter
$s_{12} = \sqrt{2 - \sqrt{4 - 1^2}}$	0.51763809	6.21165708
$s_{24} = \sqrt{2 - \sqrt{4 - (0.51763809)^2}}$	0.26105238	6.26525722
$s_{48} = \sqrt{2 - \sqrt{4 - (0.26105238)^2}}$	0.13080626	6.27870041
$s_{96} = \sqrt{2 - \sqrt{4 - (0.13080626)^2}}$	0.06543817	6.28206396
$s_{192} = \sqrt{2 - \sqrt{4 - (0.06543817)^2}}$	0.03272346	6.28290510
$s_{384} = \sqrt{2 - \sqrt{4 - (0.03272346)^2}}$	0.01636228	6.28311544
$s_{768} = \sqrt{2 - \sqrt{4 - (0.01636228)^2}}$	0.00818121	6.28316941

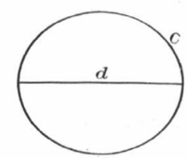
$\therefore c = 6.28317$ nearly; that is, $\pi = 3.14159$ nearly. Q.E.F.

π is an incommensurable number. We generally take

$$\pi = 3.1416, \text{ or } 3\tfrac{1}{7}, \text{ and } \frac{1}{\pi} = 0.31831.$$

PROPOSICIÓN XV. PROBLEMA

404. *Hallar la relación de la circunferencia al diámetro.*



Sean d el diámetro y C la circunferencia de un círculo.

Se desea calcular el valor de $\frac{C}{d}$, *o sea, el valor de* π.

Resolución. $2\pi r = C$ (n.° 385), y $\pi = \frac{1}{2}C$ para $r = 1$.

Sea, en general, l_n el lado de un polígono regular de n lados. Haciendo $r = 1$, se tiene también $l_6 = 1$ (n.° 394), y, aplicando la fórmula del n.° 403, se obtiene:

Fórmula	Lado	Perímetro
$l_{12} = \sqrt{2 - \sqrt{4 - 1^2}}$	0,51763809	6,21165708
$l_{24} = \sqrt{2 - \sqrt{4 - (0,51673809)^2}}$	0,26105238	6,26525722
$l_{48} = \sqrt{2 - \sqrt{4 - (0,26105238)^2}}$	0,13080626	6,27870041
$l_{96} = \sqrt{2 - \sqrt{4 - (0,13080626)^2}}$	0,06543817	6,28206396
$l_{192} = \sqrt{2 - \sqrt{4 - (0,06543817)^2}}$	0,03272346	6,28290510
$l_{384} = \sqrt{2 - \sqrt{4 - (0,03272346)^2}}$	0,01636228	6,28311544
$l_{768} = \sqrt{2 - \sqrt{4 - (0,01636228)^2}}$	0,00818121	6,28316941

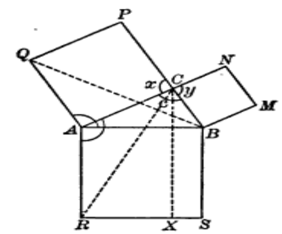
Luego, aproximadamente, $C = 6,28317$, y $\pi = 3,14159$.

La relación π es inconmensurable. El valor 3,1416 es suficientemente aproximado para casi todos los fines prácticos. Cuando no se requiere grande exactitud, $\frac{22}{7}$ es un valor cómodo.

Advanced Geometry

PROPOSITION X. THEOREM

337. *The square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides.*



Given the right triangle ABC, with AS the square on the hypotenuse, and BN, CQ the squares on the other two sides.

To prove that $AS = BN + CQ$.

Proof. Draw CX through C ∥ to BS. § 233

Draw CR and BQ.

Since ∠s c and x are rt. ∠s, the $\angle PCB$ is a straight angle, § 34

and the line PCB is a straight line. § 43

Similarly, the line ACN is a straight line.

In the △s ARC and ABQ,

$AR = AB$,

$AC = AQ$, § 65

and $\angle RAC = \angle BAQ$. Ax. 1

(Each is the sum of a rt. ∠ and the ∠ BAC.)

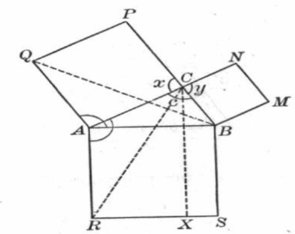
$\therefore \triangle ARC$ is congruent to $\triangle ABQ$. § 68

Furthermore the □ AX is double the $\triangle ARC$. § 325

(They have the same base AR, and the same altitude RX.)

PROPOSICIÓN X. TEOREMA

337. *El cuadrado construído sobre la hipotenusa de un triángulo rectángulo es equivalente a la suma de los cuadrados construídos sobre los catetos.*



Sean AS, BN, CQ los cuadrados construídos sobre los lados del triángulo ABC, rectángulo en C.

Demostrar que $AS = BN + CQ$.

Demostración. Trácense CX, ∥ a BS, y también CR, BQ.

Puesto que los ∠s c y x son rectos, la línea BCP es recta. N.° 43

Como $AR = AB$, $AC = AQ$, N.° 65

y $\angle RAC = \angle BAC + 1$ rt. $= \angle BAQ$, N.° 52, 1.°

los △s ARC y ABQ son iguales. N.° 68

Además □ $AX = 2\triangle ARC$. N.° 325

(Tienen una misma base AR y una misma altura RX.)

Asímismo, cuadrado $CQ = 2\triangle ABQ = 2\triangle ARC$; N.° 325

$\therefore$ □ AX es equivalente al cuadrado CQ. N.° 52, 7.°

De igual manera se demuestra que el □ BX es equivalente al cuadrado BN.

Ahora bien, $AS =$ □ $BX +$ □ AX. N.° 52, 10.°

$\therefore AS = BN + CQ$ (n.° 52, 8.°). L. C. D. D.

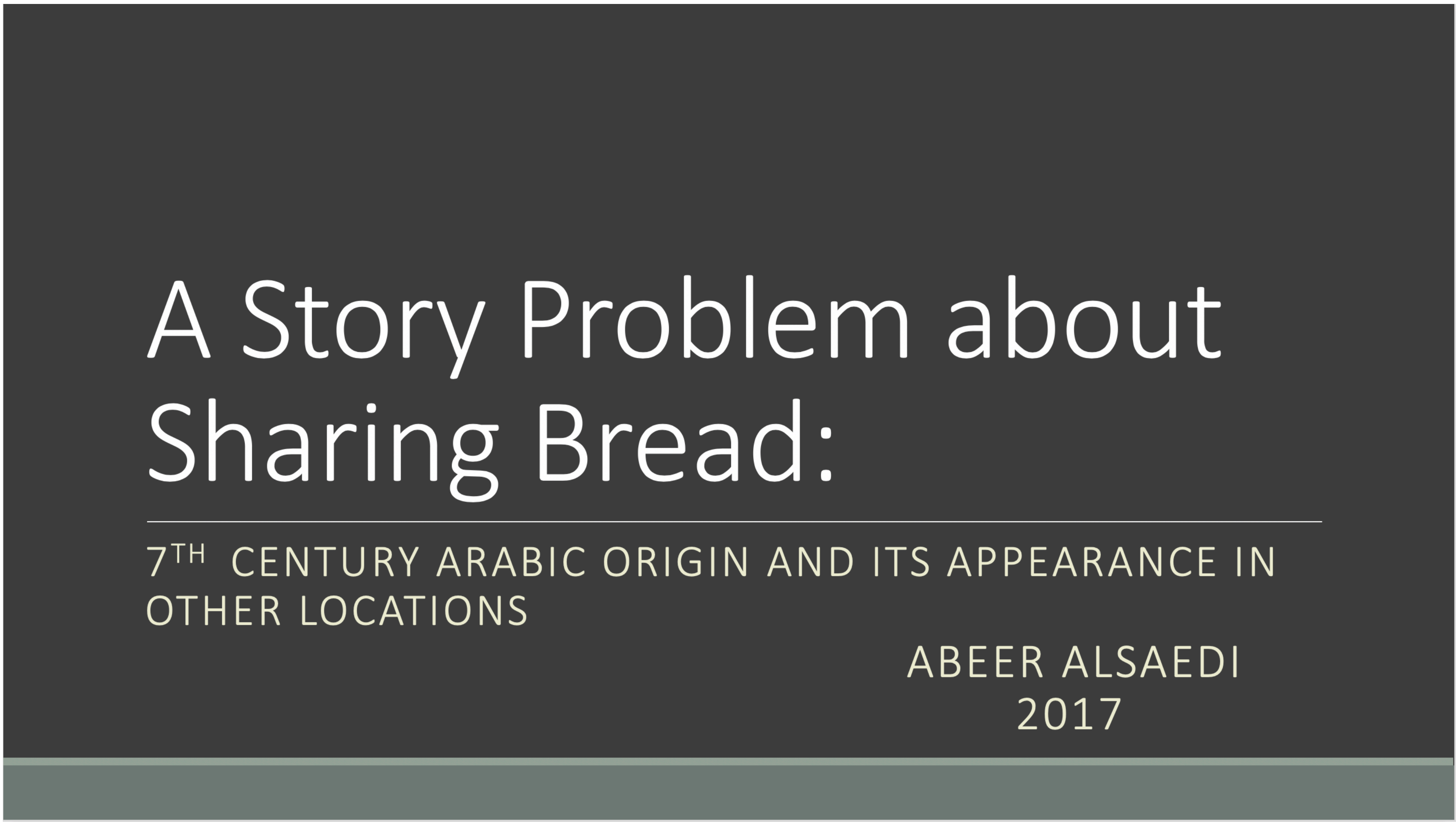
Conclusions

- Both books were printed in the United States
- The translation is effective and accommodated to the needs of the Spanish language
- The units were changed from the standard US system to the metric decimal system
- Places from the US were changed in the Spanish version
- The content on two matching pages is the same, preserving the page numbers
- Whenever the units and quantities are changed, the pedagogical content on both versions is the same
- The exercises have the same difficulty level
- Both versions have the same theorems and propositions
- The Spanish version is as good as the English version

Thank you

- Gracias

4. A final project: A Story Problem about Sharing Bread: Its 7th Century Arabic Origin and its Appearance in other Locations (Abeer Alsaedi)



Who was **Ali ibn Abi Taleb**?

- He was born in Mecca 599 AD
- He was a cousin of prophet Mohammed
- He served as a deputy and a secretary for prophet Mohammed
- He was a person of authority and high standing in the Muslim community
- He was appointed a leader by prophet Muhammad's companions in 656 AD
- He was gifted with a quick, sharp, incisive, mathematical mind:

  Dividing 17 Camels!
  Dividing an Inheritance !
  Two travelers, one with 5 and the other 3 loaves of bread ! … etc
- He was assassinated in Kufa 661 AD

The Loaves of Bread Puzzle

which seems to have first been asked of Ali-Ibn-Abi Talib, the 4th caliph of early Islam (600-661 AD).

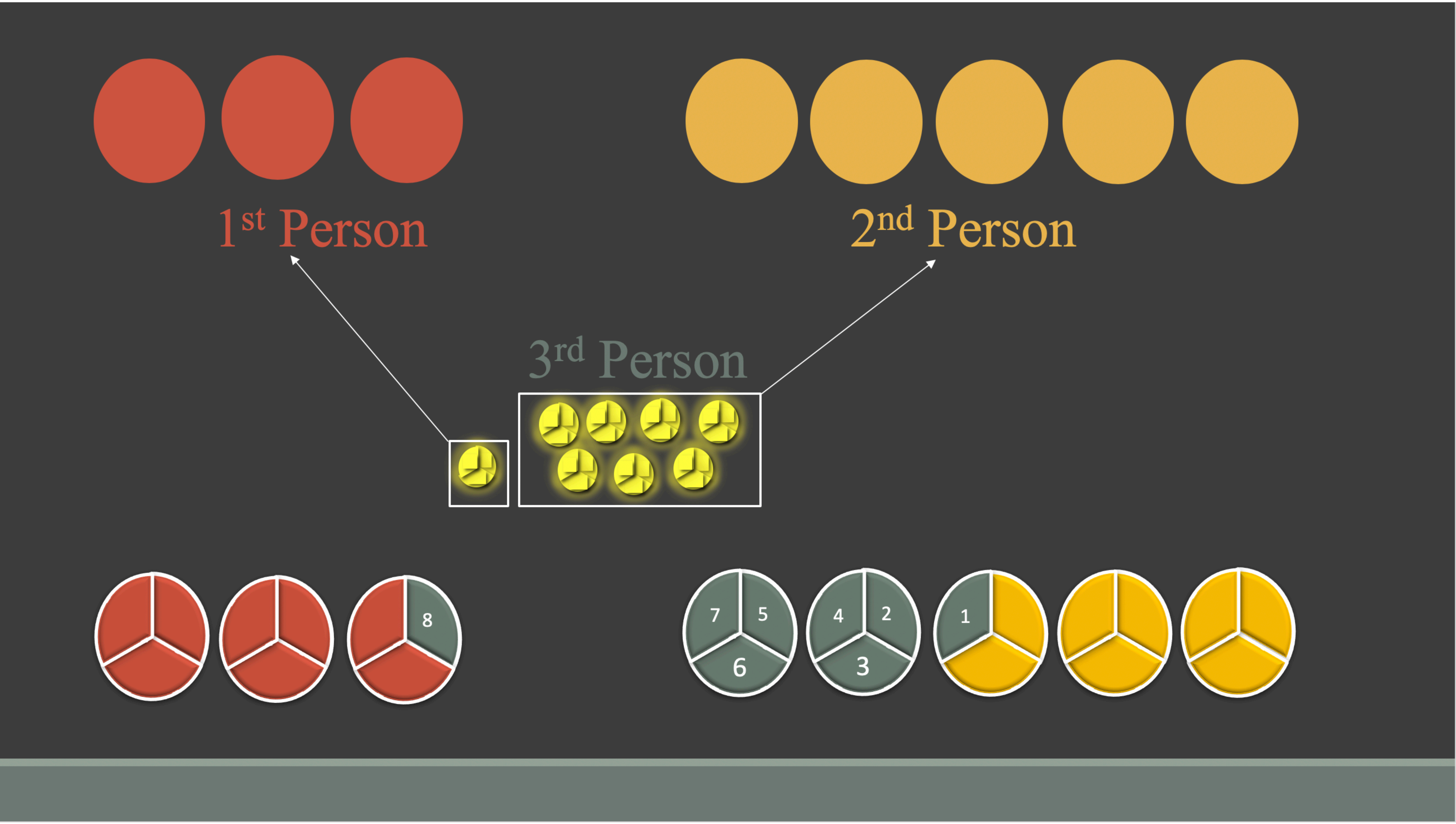
There were two men having a meal. The first man brought 5 loaves of bread, and the second brought 3. A third man, Ali, came and joined them. They together ate the whole 8 loaves. As he left, Ali gave the men 8 coins as a thank you.

The first man said he would take 5 of the coins and give his partner 3, but the second man refused and asked for half of the sum (i.e., 4 coins) as an equal division. The first man refused.

They went to Ali and asked for a fair solution. Ali told the second man (who contributed 3 loaves), “I think it is better for you to accept your partner‘s offer.“ But the man refused and asked for justice.

So Ali said, “Then I say that the man who offered five loaves takes 7 coins and the man who offered three loaves takes one coin.

Can you explain why this was actually fair?



1st Person
2nd Person
3rd Person
8
7
5
6
4
2
3
1

Names for the Puzzles

- The Partnership
- Dinner for Three
- The Three Travelers
- The Stranger's Dinner
- Three Hobos, Three Loaves, and Dollar

In terms of mathematics :

- Mathematical Recreations
- Mental Arithmetic
- Decision of dispute
- Unequal Sharing

Timeline of the Puzzle

Arabic Ali Ibn Abi Taleb 656-661	Latin Fibonacci, *“Liber Abaci”* 1202	Italian Paolo Dell’Abbaco, *“Trattato d’Aritmetica”* 1282 - 1374	Arabic Ibn Abi Hajala, *“Unmūdhaj Al-Qitāl Fī Naql Al-‘Awāl”* 1375
English Uriah Parke, *“Lectures on the Philosophy of Numbers,"* 1842	English D. Adams, *“Scholar’s Arithmetic”* 1801	Italian Calandri, *“De Arithmetica Opusculum”* 1518	Arabic As-Suyuti, *“Tareekh Ul Khulafa”* 1489
English Edward Brooks, *“The Normal Mental Arithmetic”* 1873	English S. E., Clarke, *“Mental Nuts”* 1900	English C. Arthur Pearson , *“The Royal Magazine”* 1903	English A.S. Draper & Charles Welsh, *“Self Culture for young people “* 1906
English Paul Sloane, *“Lateral Thinking Puzzlers”* 1991	English Joesph Degrazia , *“Math is Fun”* 1976	English Maurice Kraitchik, *“Mathematical Recreations”* 1943	English A. S. Draper & Charles Welsh, *“Draper's Culture for young people”* 1907

Fibonacci , Leonard

Liber Abaci (1202)

3. There were two men, of whom the first had 3 small loaves of bread and the other, 2. They walked to a spring, where they sat down and ate; and a soldier joined them and shared their meal, each of the three men eating the same amount; and when all the bread was eaten, the soldier departed, leaving 5 *bezants* to pay for his meal. The first man accepted 3 of these *bezants*, since he had had 3 loaves; the other took the remaining 2 *bezants* for his 2 loaves. Was the division fair?

    Answer: No! The man who gave 3 loaves should receive 4 *bezants*; the other man, 1 *bezant*.

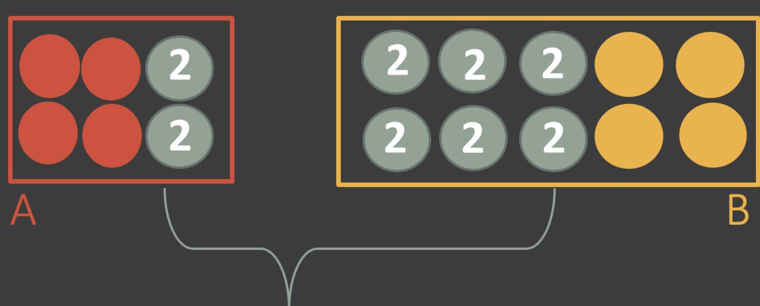
Frank J. Swetz. *"Learning Activities from the History of Mathematics"* (1993)

Edward Brooks

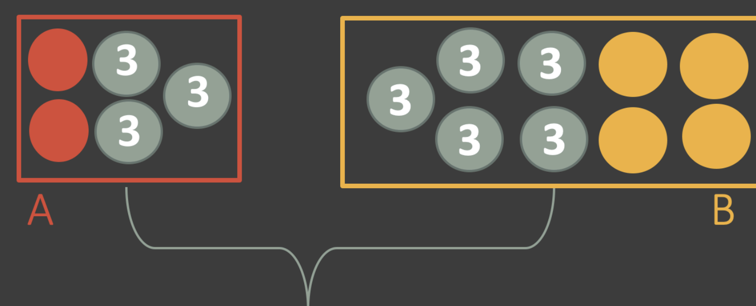
The Normal Mental Arithmetic: A Thorough and Complete Course by Analysis and Induction (1863)

15. A furnished 6 eggs for a repast, and B 10 eggs, while C contributed 16 cents to be divided between A and B; how much shall each receive, provided A and B eat the same number, and C eats 4 more than each?

17. A, B, and C eat 14 peaches, of which A owned 5 and B 9, and C contributed 24 cents; how much of the money ought A and B each to receive, if B eats twice as many as A, and C eats twice as many as B?



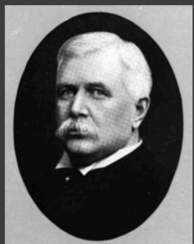
C ate 4 more than A and B



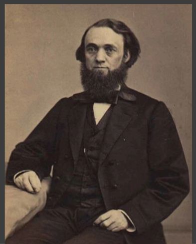
B ate twice as many as A

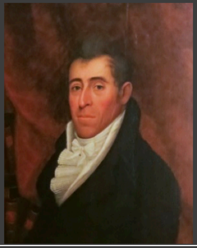
C ate twice as many as B

Conclusion



20th



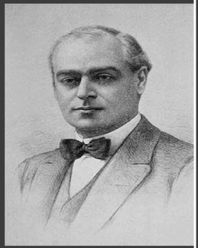


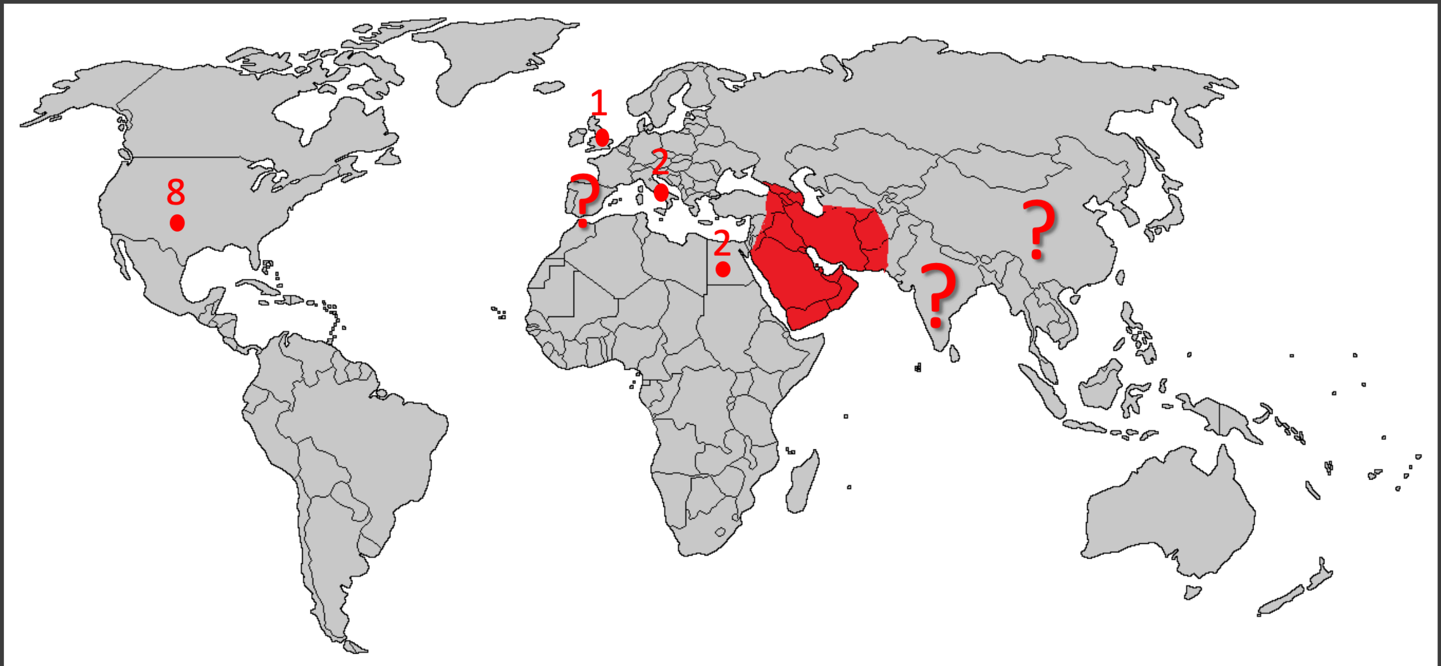
19th

16th

15th

13th





THANK YOU

Final Remarks

We have seen snippets of the work of four students who took the History and Theories of Mathematics Education course, one book report and three final projects:

- An 1870 arithmetic text in the Cherokee language (Robin Hastings)
- Is Number Stories of Long Ago (1915) by David Eugene Smith ethnomathematics? (Breanna Desbean)
- Comparing Geometria Plana y del Espacio (1915) and Plane and Solid Geometry (1913), both by G.A. Wentworth and David Eugene Smith (Jose Terrazas-Reyes)
- A story problem about sharing bread: Its 7th century Arabic origin and its appearance in other locations (Abeer Alsaedi)

This is clearly a different kind of course.

There are three aspects regarding how the material the students study is chosen:

1. The math included in the materials is not above the heads of the students.

But the math doesn‘t have to be at a very high level. There is much more to pay attention to: The historical context of the material.
For example, in the nineteenth century zero was not a number. Addition was not an operation, but something you do with numbers. And the content of each book was different, depending on the social and economic position of students in the society at the time, and what they needed to learn.

So in preparing book reports and projects, students also learn about the history of mathematics.

2. In working with the material for book reports, students become familiar with the social, cultural, and political structure of the times in which their books were written. For example, books written for women, or written during the Civil War.

It is interesting to see how students choose their books for book reports, and then, how they often go from them to their final projects.

Native English speakers often choose English texts. Other speakers e.g., Chinese/Arabic/Spanish, often choose texts in their native language.

Many Hispanic students in the class were very sensitive to the possiblity that material for Spanish-speaking children is made less challenging.

3. The personal interest of a student in the topic is needed, as projects require hours of work. A student‘s interest is strongly correlated with the product he/she produces.

Students are given choices from scores of old books, from my personal collection and from the internet.

They don‘t have complete freedom; it is a one-semester course so I have to keep them on track.

Students who want to continue to work on their projects may continue with independent study. Several have done this and their work has resulted in published articles.

The plan is that the course will be offered again in Fall 2020.

The plan is that the course will be offered again in Fall 2020.

Thank you!

ADDITION.

§ 15. 1. If you have 2 cents, and find 3 cents, how many cents will you have? Ans. 5.

2. I spent 12 cents for a slate, and 5 cents for a copy-book; how many cents did I spend? Ans. 17 cents.

3. John gave 6 cents for an orange, 7 cents for a lead pencil, and 9 cents for a ball; how many cents did they all cost? Ans. 22 cents.

§ 16. The process of uniting two or more numbers into one number is called Addition.

The number obtained by the addition is called the sum.

OF THE SIGNS.

§ 17. The sign +, called *plus*, means added. When it stands between two numbers, it shows that they are to be added. Thus: 4+2 means that 4 and 2 are to be added together.

The sign = is called the sign of equality, and denotes that the quantities between which it stands equal each other.

The expression: 4+2=6, means that the sum of 4 and 2 is 6. Read 4 and 2 are six.

ADDITION TABLE.

2 and 1 are 3	3 and 1 are 4	4 and 1 are 5
2 " 2 " 4	3 " 2 " 5	4 " 2 " 6
2 " 3 " 5	3 " 3 " 6	4 " 3 " 7
2 " 4 " 6	3 " 4 " 7	4 " 4 " 8
2 " 5 " 7	3 " 5 " 8	4 " 5 " 9
2 " 6 " 8	3 " 6 " 9	4 " 6 " 10
2 " 7 " 9	3 " 7 " 10	4 " 7 " 11
2 " 8 " 10	3 " 8 " 11	4 " 8 " 12
2 " 9 " 11	3 " 9 " 12	4 " 9 " 13
2 " 10 " 12	3 " 10 " 13	4 " 10 " 14
2 " 11 " 13	3 " 11 " 14	4 " 11 " 15
2 " 12 " 14	3 " 12 " 15	4 " 12 " 16

2. I spent 12 cents for a slate, and 5 cents for a copy book: how many cents did I spend? *Ans.* 17 cents.

From Ray's Arithmetic Third Book Practical Arithmetic by Induction and Analysis (1857)