Students' knowledge of geometric abstractions

Patricia Baggett, New Mexico State University
Andrzej Ehrenfeucht, University of Colorado

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Abstract

Students can acquire considerable skills in geometric constructions already in the early grades. But the science of geometry is based on several abstractions that cannot be deduced from direct observations. The three main abstractions are the following:

(1) Points do not have size; their diameters have a length of zero; and therefore no two points can touch.

(2) The points on a line are dense, which means that between any two different points there are many other points.

(3) A straight line is a set of points, as are other geometric figures.

Modern analytic geometry is based on a one-to-one continuous correspondence between the real numbers and points on a straight line. It allows us to assign numerical values to all geometric quantities such as length, area, and so on.

In the first study, which involved 443 subjects, we tried to answer the question of whether middle school, high school, and college students accept these three abstractions.

In the second study (a small pilot study), involving only 22 subjects, we addressed the matching between real numbers and geometric quantities.

Introduction

When we teach mathematics to students, we test them by asking questions and giving them problems to solve. We judge both their skills and their understanding based on the correctness of their answers. Correctness of answers is clearly a measure of a student's skills, but does it test a student's understanding of mathematics?

Good students know very well what is expected from them, and what answer will bring them a good grade. So is it possible that their answers do not reflect what they think about a given topic, but only reflect their prediction about which answer will bring the highest grade?

We asked students questions about mathematical topics, telling them,

"This is not a test."
"We want to know your opinion about this topic."
"We will not judge your answer as correct or incorrect."
"Don't sign your name. Your answers should be anonymous."

We started with questions in geometry, because many educators (e.g., Bass, 2001) believe that students have a good "intuitive" grasp of geometry, which forms a solid foundation that is needed for teaching the "more abstract" domains of mathematics, such as the arithmetic of real numbers or algebra.

Study 1. Geometric abstractions

Purpose of the study and the method

Modern geometry is based on three abstractions:
(1) Points do not have size; their diameters have a length of zero; and therefore no two points can touch.
(2) The points on a line are dense, which means that between any two different points there are many other points.
(3) A straight line is a set of points, as are other geometric figures.

We tested whether subjects agree or disagree with (1), (2), and (3) by asking them three of the following questions (the two answers, yes and no, in parentheses, are in agreement).

(1a) Are any two points on a straight line some distance apart? (Yes)
(1b) Do some points on a straight line touch each other? (No)

(2a) Take any two points on a straight line. Are there always some other points between them? (Yes)
(2b) Are there some points on a straight line that are so close together that there are no other points between them? (No)

(3a) Is a straight line completely filled up with points? (Yes)
(3b) Is there some empty space between points on a straight line? (No)

There were eight versions (2x2x2) of the questionnaire that were randomly distributed, and each subject answered only one of the eight.

Subjects.

We tested the following groups of subjects:

1. 6th grade	(60 subjects)	Total cohort from 1 school.
2. 6th grade	(59 subjects)	Total cohort from 2nd school.
3. High school 9th grade	(93 subjects)	All students from 1 school.
4. High school geometry	(31 subjects)	Second high school.
5. High school algebra I	(38 subjects)	Third high school.
6. High school algebra II	(43 subjects)	Third high school.
7. Community college	(24 subjects)	General math.
8. College freshmen	(39 subjects)	Elementary math for preservice teachers.
9. College juniors	(35 subjects)	CS majors. Already had lin. algebra & 2 sem. of calculus.
10. Graduate students	(21 subjects)	Grad. students and practicing teachers taking elementary math course.
Total	(443 subjects)	

Results and their interpretations

Percentage of mathematically correct answers				
Group:	**Points have no size:**	**Ordering is dense:**	**Line is a set of points:**	**All three questions:**
1. 6th grade (60)	**45 (75%)**	**24 (40%)**	**35 (58%)**	**9 (15%)**
2. 6th grade (59)	**35 (59%)**	**33 (56%)**	**28 (47%)**	**10 (17%)**
3. HS 9th grade (93)	**60 (64%)**	**44 (47%)**	**44 (47%)**	**15 (16%)**
4. HS geometry (31)	**16 (51%)**	**16 (51%)**	**17 (55%)**	**5 (16%)**
5. HS algebra I (38)	**27 (71%)**	**17 (45%)**	**17 (45%)**	**7 (18%)**
6. HS algebra II (43)	**36 (84%)**	**30 (70%)**	**30 (70%)**	**14 (32%)**
7. Community college (24)	**17 (71%)**	**16 (67%)**	**13 (54%)**	**5 (20%)**
8. College freshmen (39)	**30 (77%)**	**28 (72%)**	**27 (69%)**	**15 (38%)**
9. College juniors (35)	**27 (77%)**	**26 (74%)**	**23 (65%)**	**17 (48%)**
10. Graduate students (21)	**17 (81%)**	**17 (81%)**	**15 (71%)**	**10 (48%)**
Total (443)	**310 (70%)**	**251 (57%)**	**249 (56%)**	**107 (24%)**

We did not observe any change between the sixth and ninth grades.

Student variables such as SES and ESL, which were unevenly distributed among schools, could account for all the differences.

Algebra II students did better than community college students, and almost as well as freshmen. This is not surprising. In this school, algebra II was taken by students who planned to go to college. So this group was self-selected.

It is consistent with these data to assume that students do not change their opinion about geometric abstraction after the sixth grade. It is also consistent that by a process of selection, the percentage of students with a "standard" (Euclidean) view of geometry increases.

These data do not mean that the students would fail a geometry test. We talked to several computer science majors after they filled out the questionnaire, and some of them said that on a test they would answer differently, because they knew what answers would be judged as correct by an instructor.

Consistency of "incorrect" answers

Among the eight possible patterns of answers, six are consistent, but two are not.

For example: No, No, Yes, means that points have size, points touch each other, the ordering is not dense, each point touches its two neighbors, and so the points fill up the line.

This is an "atomistic" view of a line,

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... |_|_|_|_|_|_|_|_|_|_|_|_|_|_|_|_|_|_|_|_|_|_|_|_|_|_|_|_| ...
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Points are very small blocks, just like miniature pixels on a computer screen, but much smaller. This is a possible model of a line, but it is not the model provided by Euclidean geometry.

The two inconsistent patterns of answers are <u>No, Yes, No</u>, and <u>No, Yes, Yes</u>, because if points touch each other or just have positive diameters (size), the ordering cannot be dense. (There is no point that separates two touching points.)

Percentage of inconsistent answers		
Group:	**Its size:**	**Inconsistent answers:**
1. 6th grade	**60**	**8 (13%)**
2. 6th grade	**59**	**14 (24%)**
3. HS 9th grade	**93**	**16 (17%)**
4. HS geometry	**31**	**8 (26%)**
5. HS algebra I	**38**	**6 (16%)**
6. HS algebra II	**43**	**6 (14%)**
7. Community college	**24**	**7 (29%)**
8. College freshmen	**39**	**7 (18%)**
9. College juniors	**35**	**2 (6%)**
10. Graduate students	**21**	**2 (10%)**
Total	**443**	**76 (17%)**

Notice that 30% of subjects answered No to the first question;
57% of subjects answered Yes to the second question;
and
.30*.57 = 17% of subjects answered No, Yes to both questions.

This indicates that subjects answered the two questions independently.

This is a strong indication that most subjects did not have a specific model in mind when they answered the questionnaire. (If they had a specific model in mind, they would not answer inconsistently.) It looks as if they judged the plausibility of each answer separately, and disregarded their interdependence and the possible logical consequences.

Study 2 (a pilot study). Six square inches

This study was conducted in an undergraduate class in mathematics taken by 31 prospective and practicing teachers. The class, which is run in a laboratory format, covers topics related to elementary arithmetic and geometry taught in elementary and middle school.

Two days after the discussion of the relationship between perimeters and areas of rectangles, the students were asked the following question,
"Is there a square that has an area of exactly six square inches?"
And they were asked to justify their answers.

The students were told that it was not a test, that they should not write their names (their answers should be anonymous), and they should just say what they think. They were not even obliged to answer the question.

Twenty-two students (out of 31) answered the question ("Is there a square that has an area of exactly six square inches?"), but not all provided a justification.

Answer:	**Number of students:**	**Justification:**
yes and no	**1**	**(1) Theoretically yes, but it would be impossible to measure precisely enough to draw one.**
I don't know.	**4**	**(1) I was sick when you taught this lesson.** **(2) I did not understand the question, nor did I understand the conversation that happened in class.** **(3) I do not know, I didn't think about it. Seems if you use whole units, you can't do it. Square root of 6 = 2.4494897 (according to calculator), so maybe if square had sides of that length (draws square and labels side lengths as 2.4494897, same, same, same) l x w = area except 2.4494897 x itself = 5.9999997 so ? not perfect.** **(4) No reason given.**
yes	**5**	**(1) 1.5 x 4 = 6.0. The sides would be 1.5.**

		(2) The sides would be 1 1/2 ins. **(Drawing of a square with each side labeled 1 1/2.)** **(3 & 4) (Two students just included a drawing of a square six inches by six inches.)** **(5) No reason given.**
no	**12**	**(1) But it comes pretty darn close.** **(2) Six does not have a perfect square root. The area is determined by multiplying length by width and for a square, these have to be equal.** **(3) Because it does not come out into a perfect square.** **(4) If you find the square [root] of six, it keeps going indefinitely. you could come very close, but never have an exact area of six square inches.** **(5) No matter which way I try to arrange "6 square inches", they don't make a square. is there some way to do it w/ fractions of squares that add up to 6 square inches? I just don't know. (Includes several pictures of unsuccessful attempts.**

		(6-12) No justification (7 students).

Analysis

Only one student provided an answer that can be considered to be mathematically correct, i.e. yes and no.

Why is "yes and no" a mathematically correct answer to the question, "Is there a square that has an area of exactly six square inches?"?

In mathematical models of the plane, for any positive real number, there are squares having exactly this area. So the answer "yes" is mathematically correct.
But all practical measurements of area or length are approximate, so "exact area" doesn't make much sense, and it cannot be "guaranteed". So the answer "no" is also correct.
(The question has two answers, because it can be asking either about the mathematical model of a space or about the space itself.)

Four of five students who said "yes" committed typical errors. They confused perimeter with area, or 6 in. by 6 in. (i.e., 36 square inches) with 6 square inches.

The most interesting answers are from the majority who said "no". They seem to represent the belief that irrational numbers are "inexact" and therefore they cannot be measures of geometric quantities. Such views are logically consistent, but they are in direct contradiction with the mathematical models of a 2-D plane and of 3-D space that are provided by geometry.

This means that students may "know" mathematics, but that they do not believe it.

General conclusions

In these studies we did not test subjects (students), but we asked them to tell us what they thought. We did not want to find out if they knew the "right" answer, but we asked only what answers they considered to be right.

These studies bring some disturbing questions about the effects of teaching and learning geometry in schools.

Only a minority of the students in our study accept the basic abstractions of Euclidean geometry. The acceptance is higher among college students, but this can be the result of a selection process, and not of the influence of the math courses they take.

In the small second (pilot) study, almost all students who understood the problem flatly rejected the fundamental assumption of analytic geometry, that there is a (continuous) one-to-one correspondence between the real numbers and points on a line.

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