EXPLAINING ARITHMETIC ALGORITHMS

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1. INTRODUCTION

There are two ways of explaining arithmetic algorithms. The first way, used in schools, relates algorithms to particular applications. For example, the algorithm for addition may be explained in terms of counting the number of pencils that are bundled in groups of ten, which are kept in boxes holding ten bundles each. Here, "regrouping" can mean opening a box and taking out 3 bundles.

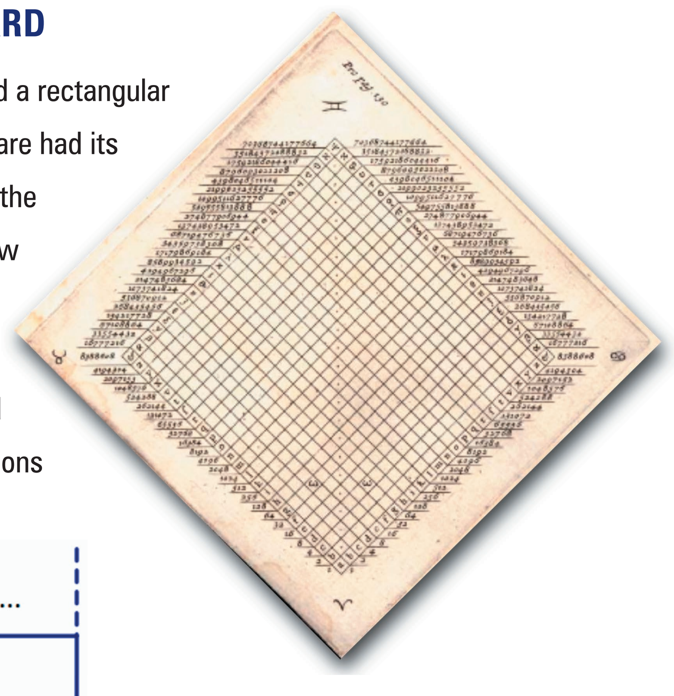
The second approach also uses concrete examples. But instead of dealing with applying numbers, the examples deal with *data that represent numbers*, such as the configuration of beads on a Japanese soroban or pebbles on a Roman abacus.

We follow the second approach and use a counting board based on John Napier's ideas, described in his *Rabdology (1617)*.

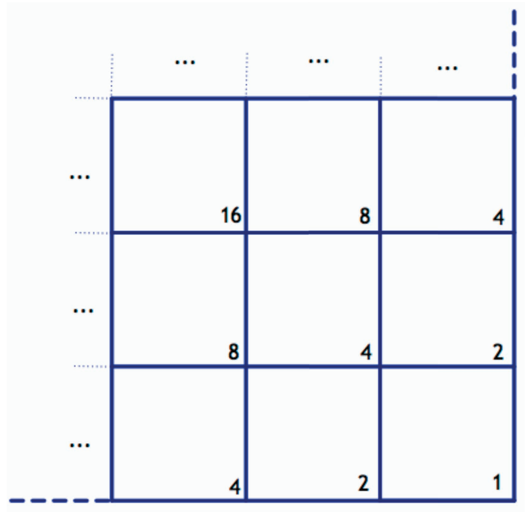
Both approaches adhere to the principle that a young learner's understanding evolves from concrete to abstract, but each approach takes a very different path. We will not address here the question of whether both approaches can be used together.

NAPIER'S COUNTING BOARD

John Napier (1550-1617) described a rectangular counting board on which each square had its local (positional) value, and where the values in each column and each row formed a geometric progression with a factor of 2. He said that the size of such a board should depend on the complexity of the computations that were being attempted.



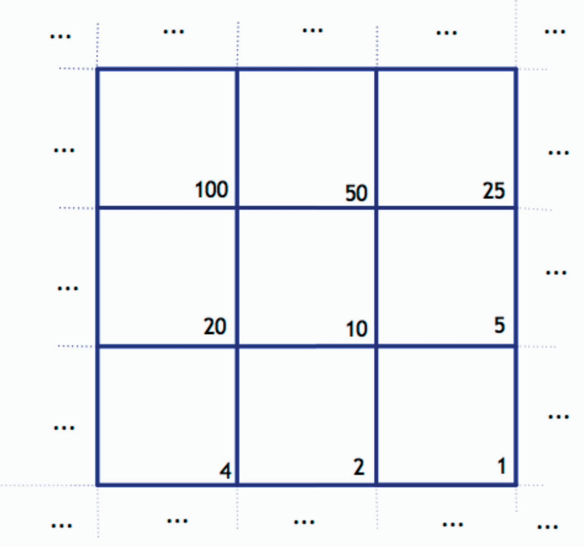
Napier's Original Board



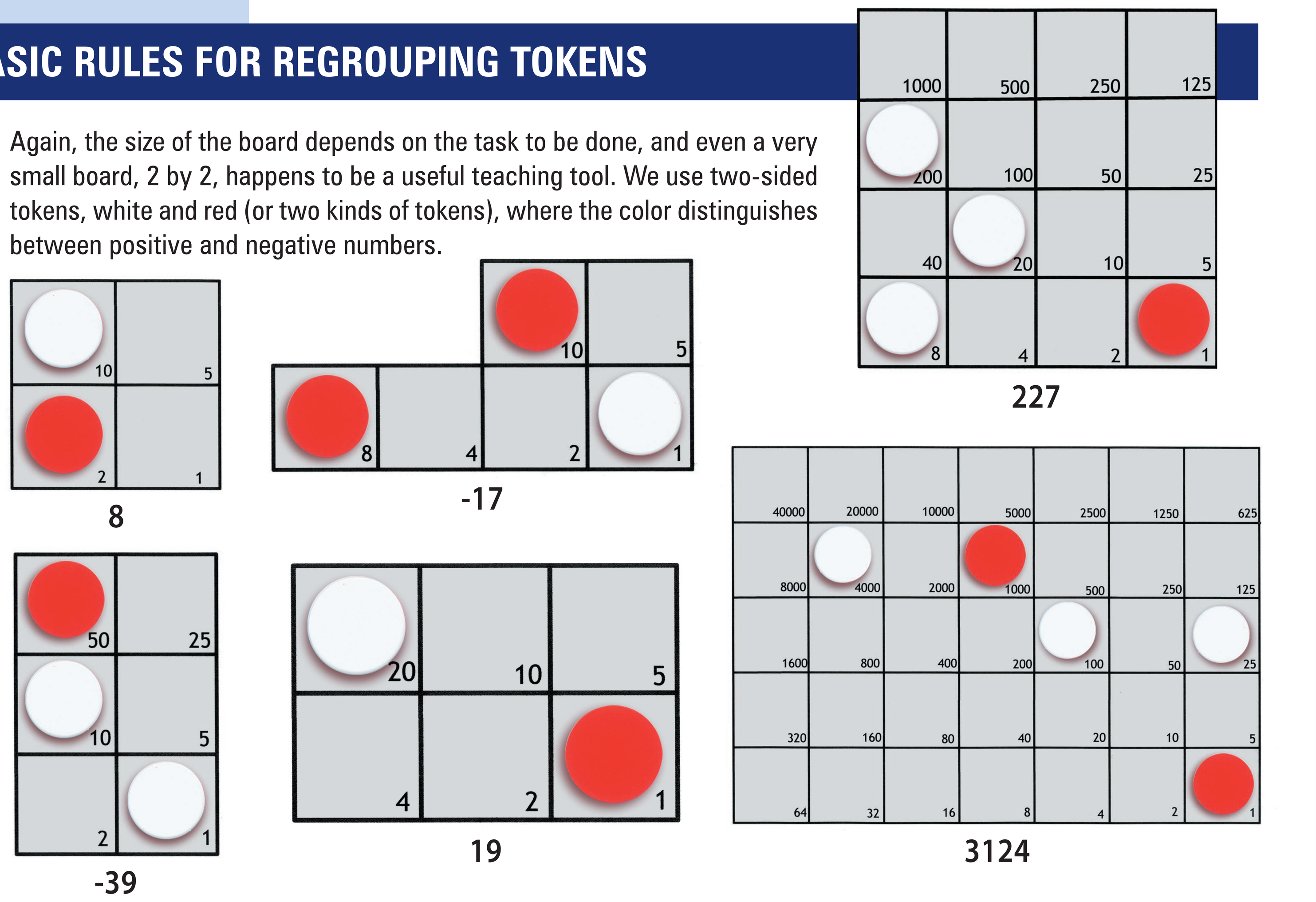
He described how to perform several arithmetic operations by using the same set of basic rules, and one kind of token (a jeton). We think that it was the first known attempt to create a universal computing device.

2. A BASE-TEN BOARD AND BASIC RULES FOR REGROUPING TOKENS

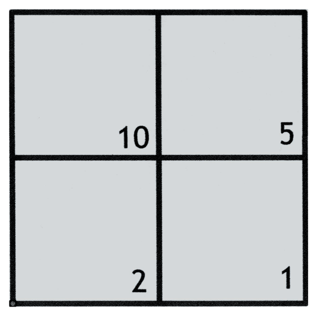
We have modified Napier's board so that the values in each row form a geometric progression with a factor of 2, but the values in each column are modified for a progression with a factor of 5.



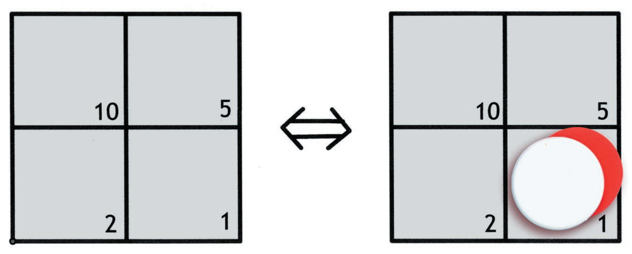
Again, the size of the board depends on the task to be done, and even a very small board, 2 by 2, happens to be a useful teaching tool. We use two-sided tokens, white and red (or two kinds of tokens), where the color distinguishes between positive and negative numbers.



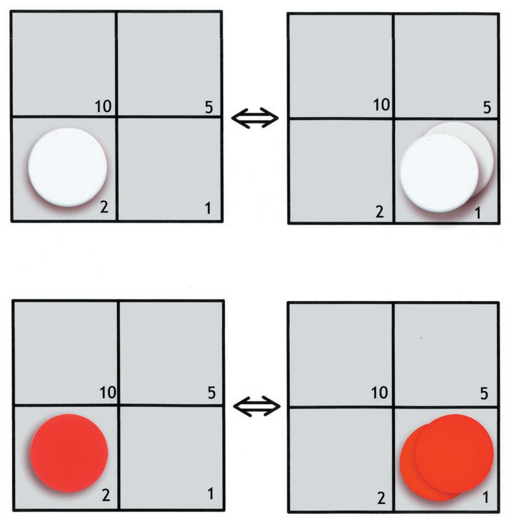
All rules for moving tokens that are already on the board have to leave the total value of the tokens unchanged. And all such rules can be obtained by composing three basic rules, which can already be demonstrated on a 2 by 2 board.



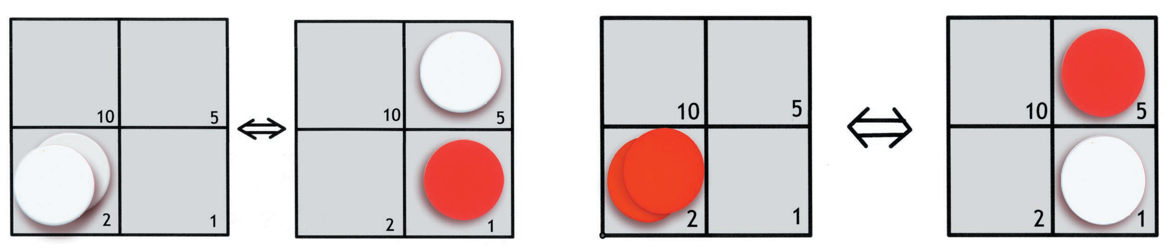
RULE 1:
Red and white tokens on the same square cancel, which corresponds to the general arithmetic fact that $x + -x = 0$.



RULE 2:
A token moved to the right has to be doubled, $2x = x + x$.



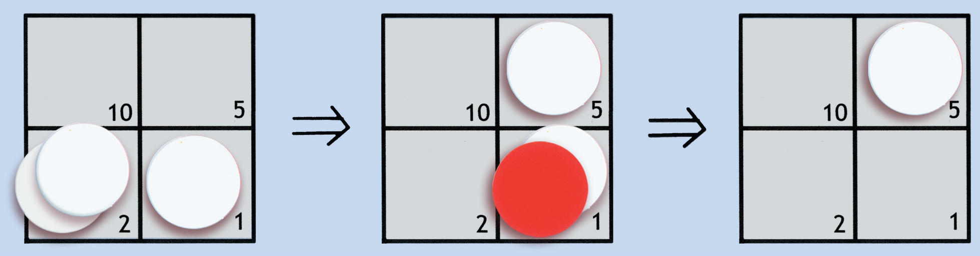
RULE 3:
This rule corresponds to the fact that $2x + 2x = 5x + -x$.



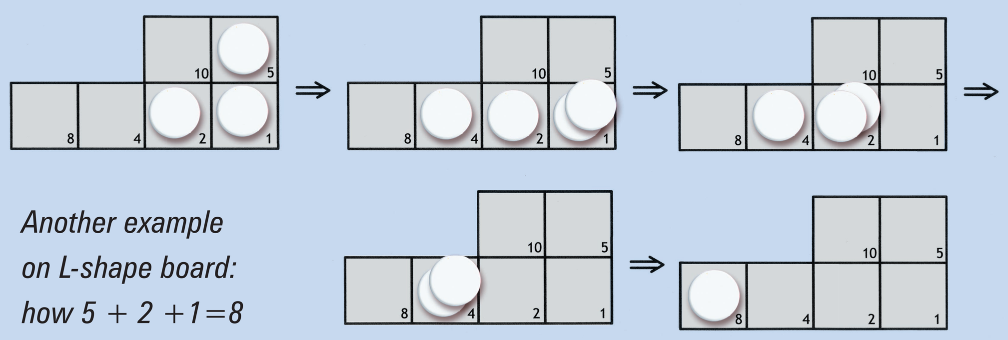
For x >0

For x <0

An important property of any board for which the values of rows and columns form a geometric progression: Any rule can be used in every place on any board. Therefore the three rules on the left are sufficient to do any regrouping on a board of any size.



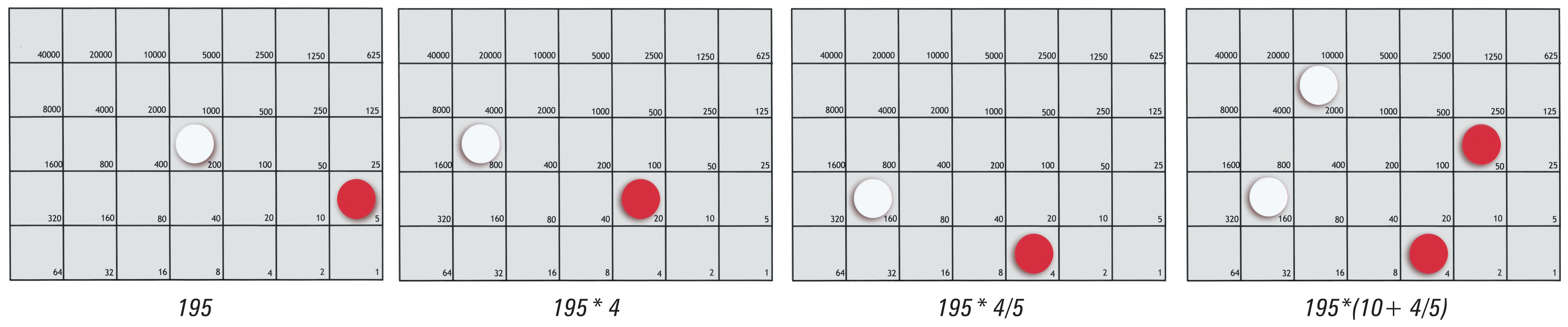
One example of deriving $2+2+1 = 5$



Another example on L-shape board: how 5 + 2 + 1=8

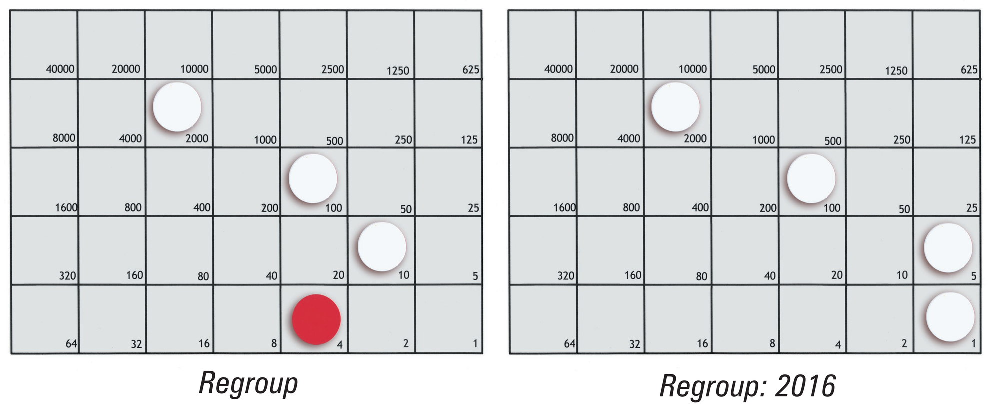
3. ARITHMETIC ALGORITHMS ON FINITE DECIMALS

Example of multiplying 195(10+ 4/5)*



School curricula require that at some time students learn algorithms for addition, subtraction, and multiplication for positive and negative numbers (integers) and decimal fractions. Also they are required to learn division of integers with remainder, and (approximate) long division with decimal fractions. These algorithms are distributed piece-by-piece among different grades, and they require different rules.

All these algorithms may be explained and executed efficiently on base-10 boards with just one set of regrouping rules, and where the size of the board depends only on the size of the numbers involved, so it can increase from grade to grade, starting with a 2 by 2 board, and ending with a board that is approximately 6 by 8.



Students do not need to know arithmetic "facts" such as, $5 + 7 = 12$ or $7*8 = 56$, to use the regrouping rules and to perform the computations. The feature that knowing addition and subtraction facts is not necessary to execute addition and subtraction, is common to many counting boards. But the feature that multiplication facts are not needed is specific to Napier's construction, in which the values of locations form geometric progressions in columns and in rows. On such boards any multiplication can be done by "shifting the multiplicand to some new positions and adding their values", which was the principle that also underlay the use of logarithms and the slide rule.

4. SUBJECTS AND PROCEDURE

Fifty-six students in two different classes, Fundamentals of elementary mathematics II and Fundamentals of elementary mathematics III (Statistics), for prospective and practicing K-8 teachers at a university in the southwest United States participated. There were 32 in the first group and 24 in the second. During one semester they were introduced to counting boards of different sizes, beginning with a 2 by 2 board and working up to a 5 by 7 board. They worked with the boards over three and a half weeks, two times per week, for 15 to 20 minutes per session. Then they filled out an anonymous questionnaire.

A SHORT DESCRIPTION OF EACH SESSION:

1. Placing numbers on a 2 by 2 board, simple addition and subtraction, regrouping
2. Learning doubling and addition on three new boards, 2 by 3, 3 by 2, and L-shaped
3. Adding ten two-digit numbers and regrouping on a 4 by 4 board
4. Adding including negative numbers on a 4 by 4 board
5. Learning multiplication by 2, 5, and 10, using two 5 by 7 boards
6. Multiplying again, this time involving distributivity
7. Multiplying negative numbers
8. Anonymous questionnaire

5. RESULTS

We report data from two groups of university students, Statistics for future teachers (24 Ss) and Elementary mathematics II for future teachers (32 Ss). First are separate opinions from each group about whether counting boards should be used in schools, and reasons not to use them. And then comments from the two groups giving reasons that they should be used, and what they learned are summarized.

GROUP: STATISTICS FOR FUTURE TEACHERS
Number of subjects: 24

Do you plan to be a teacher? yes 23, not sure 1

Have you learned anything new from playing with the boards? yes 21, no 3

Do you think that counting boards should be used in:

E - elementary, M - middle, H - high school? y - yes, n - no or no answer:

	E	M	H	no. of subjects
	y	y	y	16
	n	y	y	4
	y	y	n	5
total yes =	20	24	19	24

Reasons not to use them in early grades (4 Ss):
- [They don't teach] critical thinking.
- They are too complex.
- It may confuse children in early grades.
- No reason given (1 S)

Reasons not to use them in high school (5 Ss):
- For complex problems, using a board can be confusing.
- They would not be interesting for high school students.
- No reason given (3 Ss)

GROUP: ELEMENTARY MATHEMATICS II FOR FUTURE TEACHERS: Number of subjects: 32

Are you a teacher? yes 5

Are you planning to be a teacher? yes 25, not sure 2

Have you learned anything new from playing with the boards? yes 31, no 1

Do you think that counting boards should be used in:

E - elementary, M - middle, H - high school?
y - yes, n - no or any other answer different from "yes":

	E	M	H	no. of subjects	
	n	y	y	5	3 no, 2 other answer
	y	y	y	15	
	y	y	n	7	6 no, 1 other answer
	y	n	n	4	4 no, 4 other answer
	n	n	n	1	2 no, 1 other answer
total yes =	26	27	20	32	

Reasons why not to use them in early grades (6 Ss):
- Too difficult (2 Ss)
- Too confusing (1 S)
- No reason given (1 S)
- "Students should learn traditional algorithms first." (2 Ss)

Reasons not to use them in high school (12 Ss):
- Not useful (4 Ss)
- Too easy, not challenging enough (3 Ss)
- "Students should learn traditional math." (1 S)
- No reason given (4 Ss)

SUMMARY OF COMMENTS FROM BOTH GROUPS

1. Why should the boards be used in schools?

Most comments were general, stating that this is –

game-like, fun, visual, hands-on, interesting, easy, challenging, a different way of learning arithmetic.

More specific comments mentioned –

problem solving, critical thinking, number sense, fundamentals, learning about negative numbers, large numbers, addition, subtraction, multiplication, and not needing to learn to count, not needing to memorize.

2. What did you learn that was new by playing with the boards?

The answers were of two categories:
Specific comments, such as –

"I've learned how to add, subtract, and multiply [on the board]."

And general comments, which included –

learning new ways, different ways, new strategies for looking at mathematics,

with adjectives such as –

simple, easier, more visual, hands on.

6. CONCLUSIONS

The explanations for the standard algorithms that are taught in school meet with well known difficulties, for example, incorporating the concept of negative numbers into arithmetic, together with the infamous "Minus times minus is plus; the reason for this we shall not discuss". It is usually explained in terms of "changing direction", a concept taken from vector spaces, which are not even included in the curriculum. On a decimal board, multiplication by -1 is simply changing the color of tokens (turning them over), so doing it twice leaves the original configuration unchanged.

Also, both the skill of using the standard algorithms, and understanding them, heavily rely on a student's knowledge of arithmetic facts, and many students never fully master them. Algorithms implemented on a base-ten board do not depend on these skills, so they may be a good replacement for standard algorithms. (We do not suggest that memorizing "facts" should not be required, but only that they are not necessary prerequisites for learning arithmetic algorithms.)

So these and many other difficulties in teaching algorithms (e. g., how multiplication by a fraction can yield a result smaller than the original value), could disappear if the way algorithms are taught in schools were based on the ingenious idea of John Napier.

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