History and Theories of Mathematics Education:
A GRADUATE MATHEMATICS COURSE
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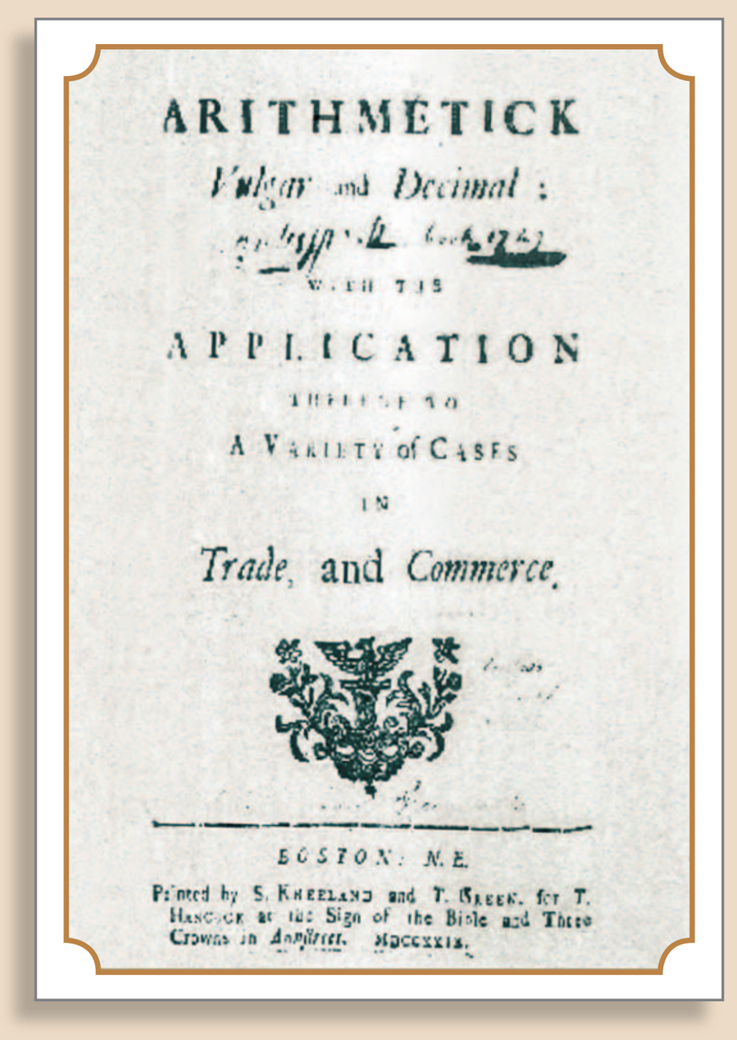
Patricia Baggett, New Mexico State University, Las Cruces, NM 88003-8001, USA, baggett@nmsu.edu
Andrzej Ehrenfeucht, University of Colorado, Boulder, CO 80309-0430, USA, andrzej.ehrenfeucht@colorado.edu

EXAMPLES OF BOOKS TO BE REVIEWED BY STUDENTS

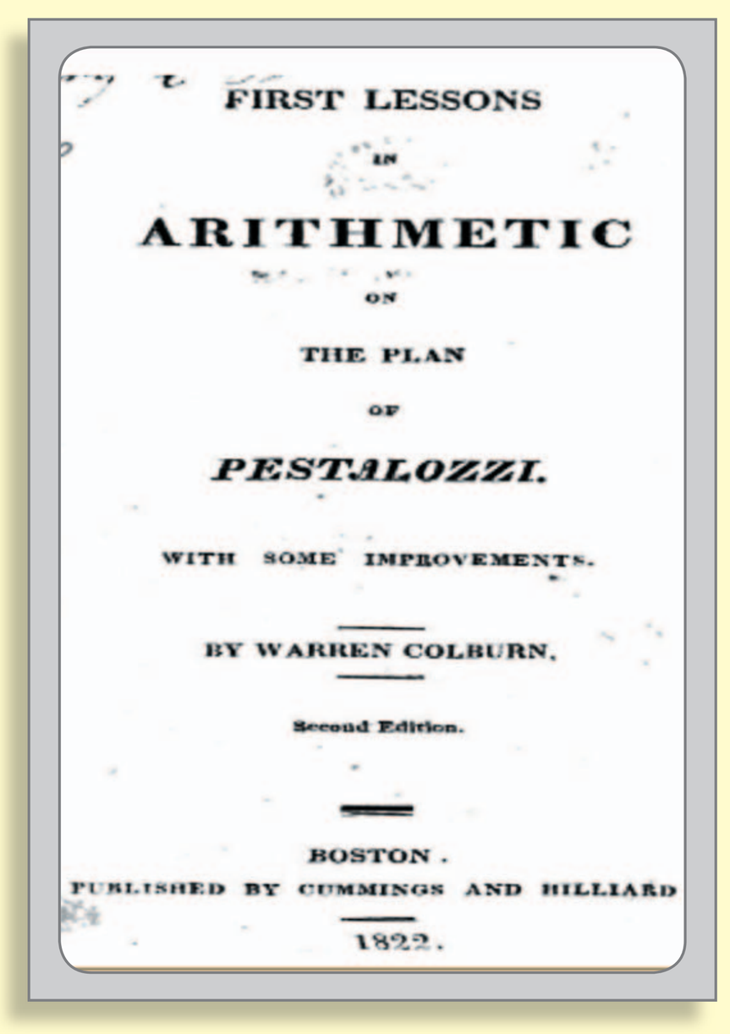


1729. Isaac Greenwood, *Arithmetick Vulgar and Decimal* (first arithmetic book by a person born in the Colonies)

p. 131.
In mixing 3 Sorts of Tobacco; the First worth 18 d. the Second 22 d. the Third 24 d. the Pound; How much of each Sort is to be taken, that the Mixture may be sold at 20 d. per Pound?
Answer 6, 2, 2

This problem has many solutions. The author describes one method of getting a "practically reasonable, and computationally easy, answer.

Notice the subject matter: tobacco!



1822. Warren Colburn, *First Lessons in Arithmetic on the Plan of Pestalozzi*

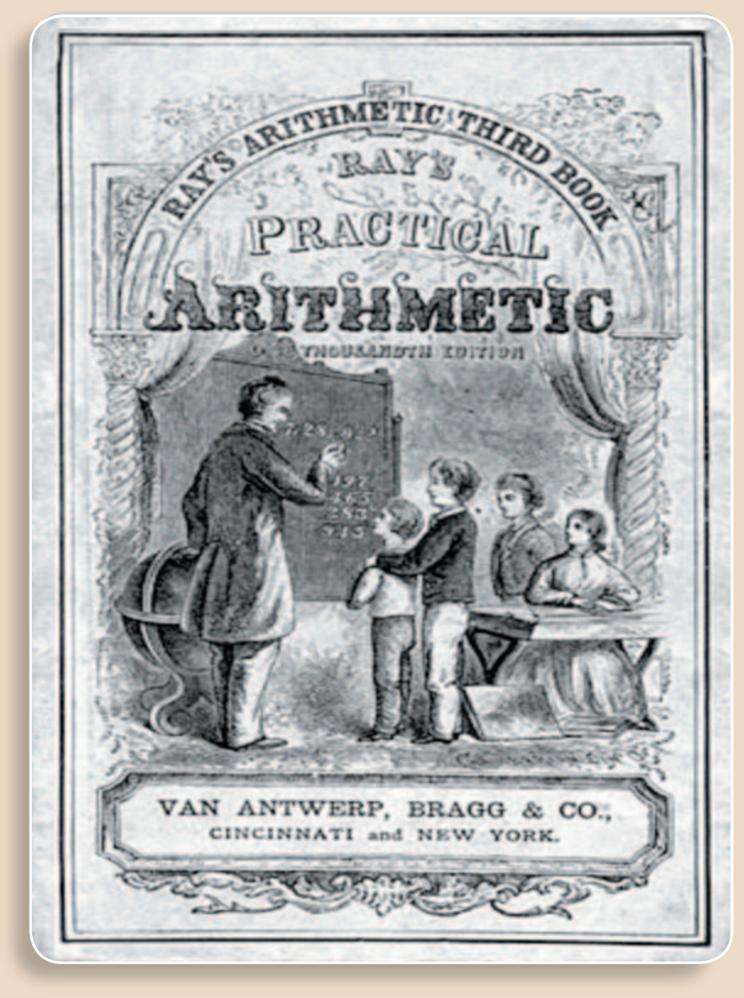
Warren Colburn's First Lessons in Arithmetic on the Plan of Pestalozzi (1822) lowered the beginning age of math instruction from the early teens to about 6 or 7.

Notice that some reader wrote the answers to the problems!

PART I.

SECTION I.

A.* 1. How many thumbs have you on your right hand? how many on your left? how many on both together? 2
2. How many hands have you? 2
3. If you have two nuts in one hand, and one in the other, how many have you in both? 3



1857 Joseph Ray, *Ray's Practical Arithmetic Third Book*

p. 96
Wine or liquid measure is used in measuring all liquids except beer, milk, and ale.

4 gills (gi)....make....	1 pint,.... marked pt
2 pints....................	1 quart,............ qt
4 quarts..................	1 gallon,........... gal
63 gallons...............	1 hogshead,...... hhd
4 hogsheads...........	1 tun,................ T.
311/2 gallons..........	1 barrel,............ bl
42 gallons...............	1 tierce,............ tr
84 gallons...............	1 puncheon,...... pu
126 gallons.............	1 pipe,............... p

Memorizing tables of units was an important part of learning. The amount of material of this kind was huge because of the large variety of regional standards and the many different units used for different produce. Also rote learning was considered to be the primary way of becoming educated. You can find an 8-volume set of Ray's freshly printed *New Practical Arithmetics* on Amazon for $79.89. They were first published in the mid 1800's.

At the Playground

When Betty and Bobby and their cousins were tired of playing school, they went to the playground. Some other children were there, too.

1. 5 boys were playing with a football, and 4 boys were playing on the slides. How many boys were playing?

2. There were 5 girls and 9 boys on the playground. How many more boys than girls were there?

106

1933
J.W. Studebaker, W.C. Findley, F. B. Knight, and W.S. Gray, *Number Stories Book Two.*

p. 106.
Here, a picture is a visual representation of the two word problems. For a while, pictures replaced manipulatives, but now both are used in parallel.

Mathematics for the Elementary School
Book 3
Teacher's Commentary, Part I
REVISED EDITION

Prepared under the supervision of the Panel on Elementary School Mathematics of the School Mathematics Study Group:

Leslie Beatty	Chula Vista City School District, Chula Vista, California
E. Glenadine Gibb	State College of Iowa
William T. Guy	University of Texas
Stanley B. Jackson	University of Maryland
Irene Sauble	Detroit Public Schools
Marshall H. Stone	University of Chicago
J. Fred Weaver	Boston University
Raymond L. Wilder	University of Michigan

New Haven and London, Yale University Press, 1965

1965
School Mathematics Study Group (SMSG) ("New Math")
Mathematics for the Elementary School grades K to 3

p. 96
A set is simply a collection of things. The things belonging to a set are called its members. A set may be defined by some property common to its members (the set of all books on this shelf; the set of all whole numbers). Sets may also consist of quite unrelated objects and may be define by simply listing their members, ...

The "New Math" considered *sets* as the most basic mathematical notion, not *numbers*.

INTRODUCTION

We offer a course, History and Theories of Mathematics Education, that has been taught twice in the Math Department at New Mexico State University, Las Cruces, NM, USA. It focuses primarily on the history of mathematics education in Colonial America and the United States, together with its theoretical foundations. We study authentic original textbooks (actual antiquarian books from a private collection, as well as texts available via Googlebooks), and look at the changes in their content and approach to the subject over time, along with writings of people who have influenced the development and changes in mathematics education. The class is in a seminar format.

A student's first assignment is a book report. Students choose a textbook from GoogleBooks and/or from antiquarian math books in the instructor's private collection, and give a 30-60 minute talk about it. Some books that have been reviewed are shown in the left panel.

Students need to acquire a lot of background information about the American Colonies, the Revolutionary War, philosophies of teaching/learning, the beginning of high schools, and other topics that come out from reading old texts. They are given reading assignments and reading questions for topics discussed in class. Sample readings are listed at the bottom of the poster.

The students' main assignment is a major research project, to be presented at a mini-conference open to faculty and students at the end of the semester. Some of these are listed in the right panel. Projects from the course have been presented at national math meetings, and one resulted in a master's thesis in mathematics.

A BRIEF TIMELINE
with events related to mathematics education:

1600s.	Colonial mathematics; Cocker's *Arithmetick* (~1678) Hornbooks Decimal fractions, symbols for operations, slide rule.
1700.	Vocational arithmetic (Daboll, Ostrander, Hawney)
1800.	Arithmetic becomes a part of general education. American arithmetic begins. Reform of Johann Pestalozzi Faculty psychology
1820.	Warren Colburn's book, based on Pestalozzi
1840.	DeMorgan's survey of arithmetic books
1850.	Three-level textbooks (e.g., Joseph Ray) Rudiments of arithmetic Intellectual arithmetic Practical arithmetic
1880.	Graded textbooks Textbooks for children Worksheets
1890.	John Dewey (progressive movement)
1910.	Peak of skills in paper and pencil computation Edward Thorndike
1960.	New Math
1970.	Elaborate multi-author books for each grade Visual (pictorial) teaching Calculators and computers
1989.	NCTM Standards Expanded role of technology Constructivism, situated cognition.
1990+.	Cognitive neuropsychological approaches to mathematical cognition
2008+.	National Mathematics Advisory Panel Report Modern issues, e.g., NCLB, Calculus reform, AP calculus International perspectives (PISA, TIMSS)
2009	Race to the top
2010+	Common Core Standards

SAMPLE READINGS

Cohen, Patricia Cline (2003). Numeracy in Nineteenth Century America, in Stanic, G. (Ed.). *A History of School Mathematics* vol. 1, p. 43-76.

Dossey, J., Halvorsen, K., & McCrone, S. (2008). *Mathematics Education in the United States*. Reston, VA: NCTM. (written for ICME 11).

Ellerton, N., & Clements, M.A. (2008). The process of decolonizing school mathematics textbooks and curricula in the United States. ICME 11, TSG 38, pp. 1-5.

Thorndike, Edward L. (1922). *The Psychology of Arithmetic*. New York: Macmillan.

FINAL REMARKS

The course will be offered again in Fall 2012 with this catalog description:

"History and theories of mathematics education. A study of the history of the mathematics taught in American schools, including an examination of authentic original textbooks and the changes in their content and approach over time, together with writings of people who have influenced the development and changes of mathematics education. Theories of learning mathematics, and current issues in mathematics education."

We hope that the historical perspective that this course offers provides students with a better understanding of the current issues that are faced in mathematics education.

EXAMPLES OF PROJECTS BY STUDENTS

A PRIEST, A RABBI, AND OBSCURE MEASURING INSTRUMENTS
(project by Kristina Leifeste)

In this project, the history and uses of two instruments, the diagonal scale and Gunter's chain, were traced.

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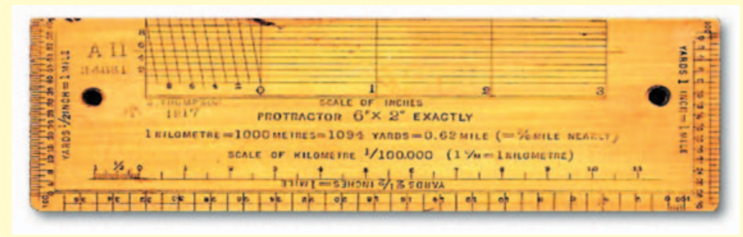
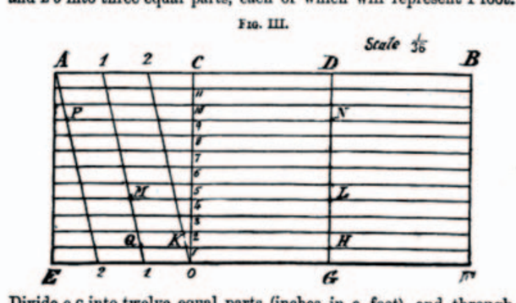


A decimal diagonal scale from the 1800's

Ex. 2. To draw a scale of $\frac{1}{36}$, showing yards, feet, and inches by diagonal division.—One inch will represent a yard. Draw a straight line AB (Fig. III.), and from A in it set off AC, CD, DB, each equal to an inch. Draw AE, EF, Co, DG, and BF, as in the last. Divide AC and Eo into three equal parts, each of which will represent 1 foot.



Divide oC into twelve equal parts (inches in a foot), and through each point of division draw lines parallel to AB. Join 2A, 11, and o2. Then in EF the distance Go=1 yard; G1=1 yard 1 foot, &c. Also the distance HK=1 yard 1 inch; HQ 1 yard 1 foot 1 inch; LM=1 yard 1 foot 4 inches; NP 1 yard 2 feet 9 inches, &c.

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A diagonal scale from The Art Student's Second Grade Practical Geometry – John Lowres, London (1876)

The diagonal scale is a two-dimensional scale drawn or etched on a flat medium for the purpost of measuring lengths precisely and accurately to three figures. It has been featured in mathematics textbooks from 1714 to the present day.

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Gunter's chain was developed in 1620 by Edmund Gunter (1581-1626), an English clergyman and mathematician.

It was used for measuring and surveying land for large commercial purposes. The chain was divided into 100 links, separated into groups of 10 links by brass rings. Each link was 7.92 inches long. The entire chain is 22 yards long (the length of a cricket field), or 66 feet, or 0.0125 miles.

ZERO IS A TROUBLEMAKER
(project by Deepak Basyal)

In this project, the history of zero was traced, especially concentrating on its treatment in books used in the Americas from Edward Cocke (first edition, 1676; 1719) through the "New Math" period.

Name for Zero or Nothing	Who Used It and When
sunya	Means "empty"; used in India around 400 A.D. to indicate an empty column on a counting board.
nulla	Latin word for "nothing"; used by a monk in Rome, Italy, around 825 A.D.
sifr	The Arab word based on the Indian word *sunya*. First used around 825 A.D.
kha	Arab word first used around 825 A.D. to mean "place" in a place-value number system. Later sometimes used to mean "zero."
nought, or naught, or ought	Old English words for "nothing." In use before 1150.
galgal	Means "zero"; first used around 1150 by a rabbi who spent most of his life in Spain. He had read Arab math books.
zephirum	The Latin word based on the Arab word *sifr*. First used in 1202 by an Italian merchant.
cipher or cypher	An English word based on the Arab word *sifr*. First used between 1350 and 1400.
zero	An English word based on the Arab word *sifr*. First used in a book published in 1604.

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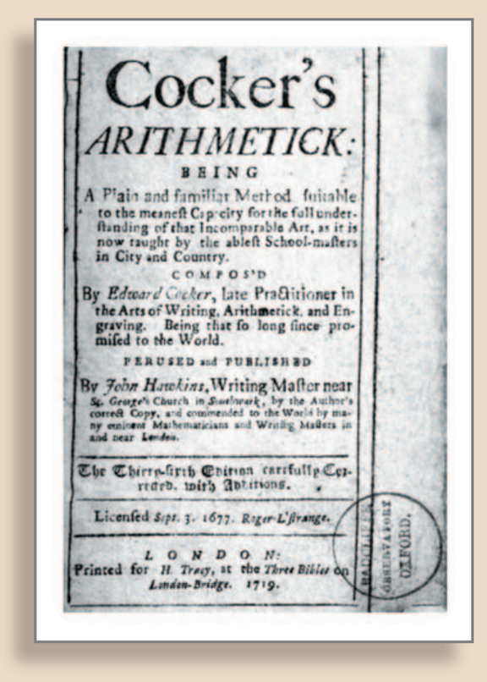
From Downey, T. (2004). The History of Zero: Exploring our Place-Value Number System



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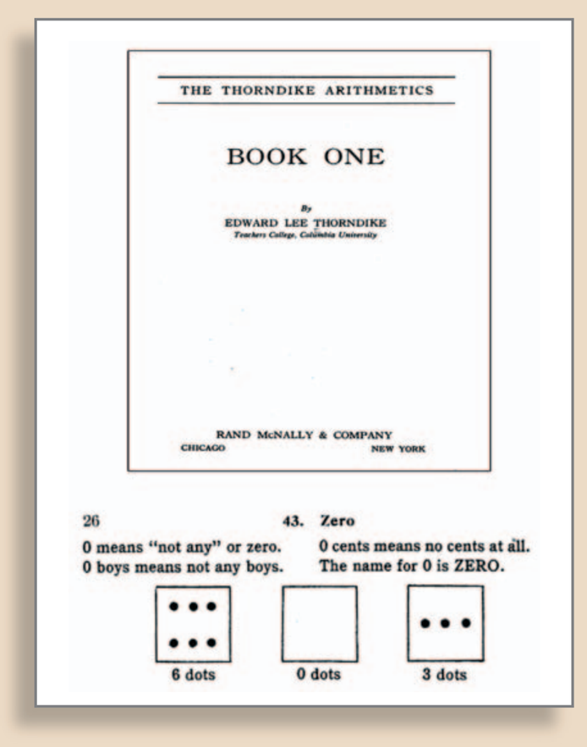
In Cocker, "Cypher ...is itself no number... cannot be divided or increas'd into Parts... so many Cyphers, though infinite in number, do make no number...


THE THORNDIKE ARITHMETICS
BOOK ONE
By
EDWARD LEE THORNDIKE
Teachers College, Columbia University
RAND McNALLY & COMPANY
CHICAGO NEW YORK
26
0 means "not any" or zero.
0 boys means not any boys.
43. Zero
0 cents means no cents at all.
The name for 0 is ZERO.
6 dots
0 dots
3 dots

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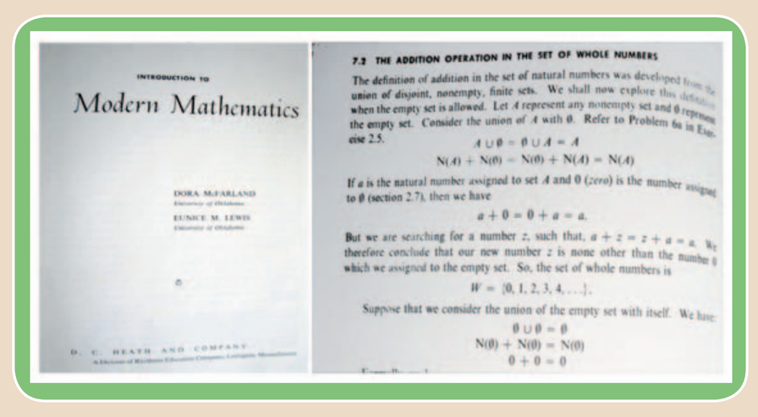
E.L. Thorndike (1917) introduced zero in his Book One.



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In the "New Math" period, zero was defined as the number of elements in the empty set. From D. McFarland and Lewis, E. (1966). Modern Mathematics.

A PRIVATEER'S SAFE PASSAGE: CALCULUS TO THE RESCUE
(project by Bryan Sims)

Beginning in 1854, and lasting until 1922, a "story" problem about a privateer appeared in American calculus books by six authors. A "privateer" is the term applied to a privately owned armed vessel whose owners are commissioned by a hostile nation to carry on naval warfare.

The calculus problem is about a privateer attempting to sail unnoticed between two lights armed vessel whose owners are commissioned by a hostile nation to carry on naval warfare.

A mathematical problem regarding the smallest illumination was, and is, very common in calculus texts. And it is usually presented without involving any stories. For example, in *Differential and Integral Calculus* by Richard Courant, 1934, p. 166, we see:
6. Two sources of light of intensities a and b are at a distance d apart. At which point of the line joining them is the illumination least? (Assume that the illumination is proportional to the intensity and inversely proportional to the square of the distance.)

9. A privateer wishes to get to sea unobserved, but has to pass between two lights A and B, on opposite head lands. The intensity of the lights is known and the distance between them. At what point between them must she pass so as to be as little in the light as possible.
Let a = the given distance AB; b = the intensity of the light A at the unit of distance, c = the intensity of B at the same distance. Let x = the distance of the point sought from A. Then since, by the principles of Optics, the intensities are inversely as the squares of the distances, we shall have for the illumination or intensity of light upon the point sought

$$\frac{b}{x^2}+\frac{c}{(a-x)^2}$$

which by the question should be a minimum.

Ans. $x=\frac{ab^{\frac{1}{3}}}{b^{\frac{1}{3}}+c^{\frac{1}{3}}}$

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The calculus problem is about a privateer attempting to sail unnoticed between two lights on opposite shores. Here is the first occurrence, from Smyth, William (1854), Elements of the Differential and Integral Calculus, Portland, ME: Sanborn & Carter:

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A Confederate privateer ship, from http://www.google.com/imgres?q=privateer+ship+during+confederacy&hl=en&biw=1024&bih=595&tbm=isch&tbnid=3OLNOpWIWYLcOM

Privateering was explicitly abolished by the Declaration of Paris of 1856, but there was a resurgence of privateering in the confederacy during the American Civil War. We hypothesize that this resurgence was the reason for the appearance of the context of privateering for finding the point on the line between lights of two different intensities at which the illumination is least.

In the project a linguistic analysis of the problem by the six authors was conducted, and we determined that in general, the texts of any two of the six authors differ in enough details for each author to claim that the problem was not copied, but just adapted from previous sources.

FROM NATURE TO PLASTIC: THE EVOLUTION OF THE USE OF CONCRETE OBJECTS IN THE TEACHING OF MATHEMATICS IN THE UNITED STATES
(project by Eva Rivera)

This project traced the use of concrete objects, beginning in vocational schools (~1800) and then in student-centered schools, from Warren Colburn (1793-1883) and John Fowle (1795-1865) to Unifix cubes (~1965).

Taking fifteen counters, place them thus: [counters arranged in rows of 1, 2, 3, 4, 5] or thus: [counters arranged in a triangle]
Then ask the child to tell which line contains 2, 1, 5, 3, or 4 counters. Then alter the position, and, putting 5 in the 3's place, and 4 in the 2's, ask which line contains 1, 3, 4, 2, 5.
Then point to a row, and ask how many times 1 are in it.

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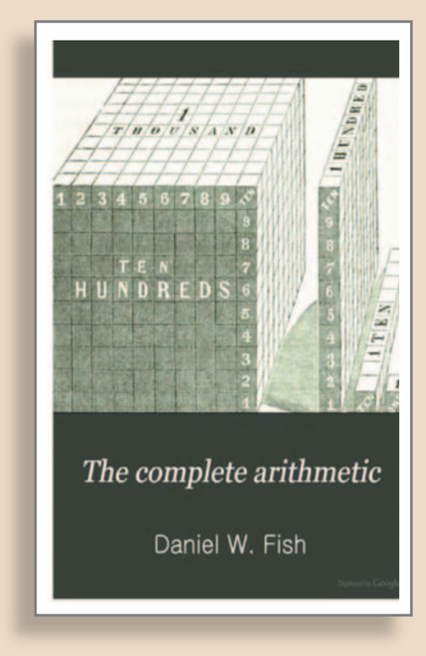
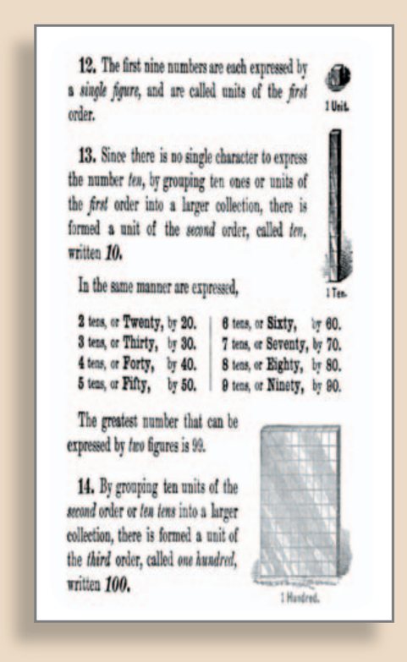
From William Fowle (1844)




12. The first nine numbers are each expressed by a single figure, and are called units of the first order.
13. Since there is no single character to express the number ten, by grouping ten ones or units of the first order into a larger collection, there is formed a unit of the second order, called ten, written 10.
14. By grouping ten units of the second order or ten tens into a larger collection, there is formed a unit of the third order, called one hundred, written 100.

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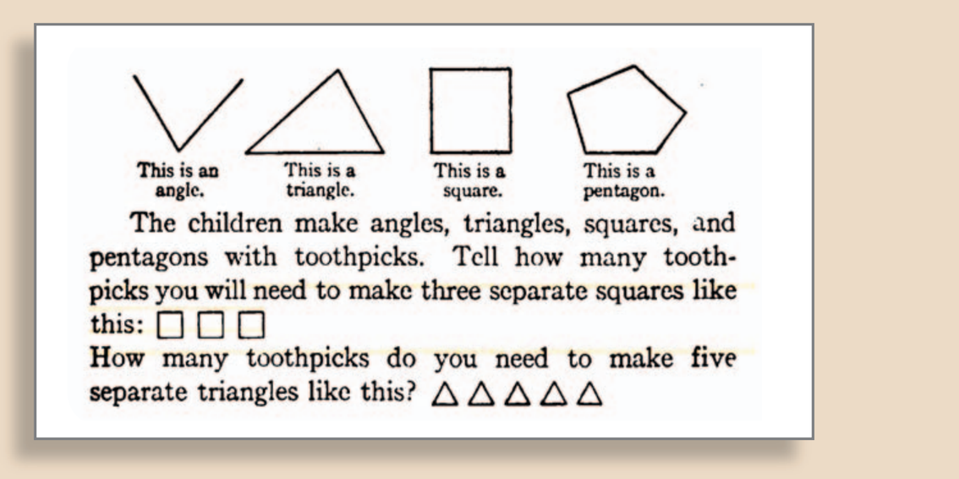
Daniel Fish (1820-1899) was the first person to introduce base ten blocks to the American curriculum.


This is an angle.
This is a triangle.
This is a square.
This is a pentagon.
The children make angles, triangles, squares, and pentagons with toothpicks. Tell how many toothpicks you will need to make three separate squares like this:
How many toothpicks do you need to make five separate triangles like this?

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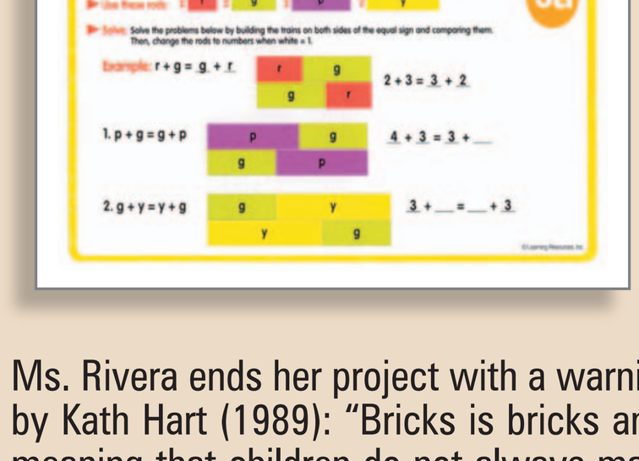
Edward Thorndike also used manipulatives. Here is an example from his 1917 book.



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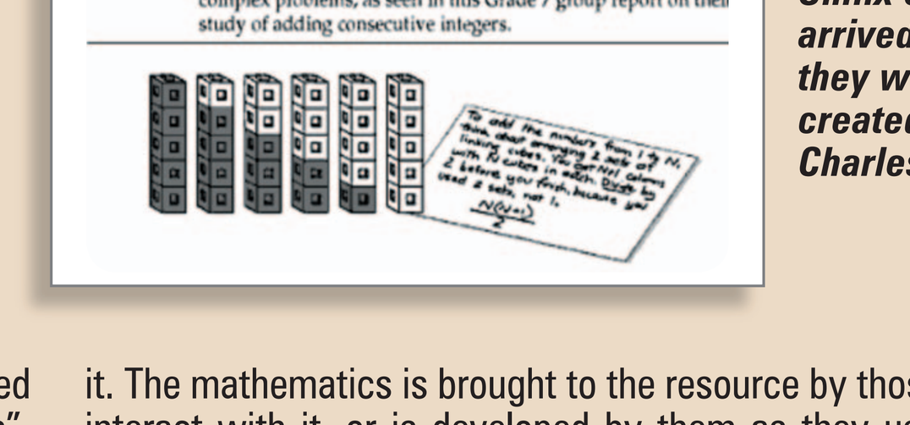
Cuisinaire rods were created by Georges Cuisinaire in 1952.



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Unifix cubes arrived in 1953; they were created by Charles Tacey.

Ms. Rivera ends her project with a warning, first verbalized by Kath Hart (1989): "Bricks is bricks and sums is sums", meaning that children do not always make the connection between concrete objects and the mathematics they are meant to represent.

And finally: An inanimate resource cannot have any miraculous power beyond the teacher and pupil who use it. The mathematics is brought to the resource by those who interact with it, or is developed by them as they use it to support or challenge their thinking (Delaney, 2001, p. 124).

Manipulatives alone cannot – and should not – be expected to carry the burden of the many problems we face in improving mathematics education in this country" (Ball, 1992, p. 46).