Contradictions in the Common Core State Standards: Problems and a Possible Solution

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Outline

1. Introduction

The Common Core State Standards in mathematics are about to become a blueprint for mathematics education in grades K to 12. They are being evaluated from many perspectives. But one point of view is missing from the discussion: Their mathematical content. This has probably happened because these new standards do not propose any significant changes in the existing curricula. They only propose a more uniform way of arranging the material that is currently taught, and highlight the topics that will be covered on standardized tests.

But the existing way of teaching mathematics in grades K to 12 is inconsistent and contains many logical contradictions that are incorporated in the Standards. Thus the proposed national curriculum misses the opportunity to correct the existing errors.

2. Sources of contradictions

The main source of inconsistency and of logical contradictions in existing curricula is the way that the number system is developed through the grades.

Page 58 from the Core Standards:

"Numbers and Number Systems.

During the years from kindergarten through eighth grade, students must repeatedly extend their conception of number.

At first, “number” means “counting number”: 1, 2, 3... Soon after that,
0 is used to represent “none” and the whole numbers are formed by the counting numbers together with zero.

The next extension is fractions. ... [Non negative rational numbers]

During middle school, fractions are augmented by negative fractions to form the rational numbers.

In Grade 8, students extend this system once more, augmenting the rational numbers with the irrational numbers to form the real numbers.

In high school, students will be exposed to yet another extension of number, when the real numbers are augmented by the imaginary numbers to form the complex numbers.

With each extension of number, the meanings of addition, subtraction, multiplication, and division are extended. In each new number system, integers, rational numbers, real numbers, and complex numbers, the four operations stay the same in two important ways:
They have the commutative, associative, and distributive properties **and their new meanings are consistent with their previous meanings**.”

But the last statement, **“and their new meanings are consistent with their previous meanings”**, is simply false. Each extension of the number system in a chain, Whole → Fractions → Rational → Real → Complex, preserves only some properties, and changes others. So some theorems which are true for whole numbers become false for fractions, and so on.

The table below shows a small sample of such properties.

Number systems:

Statement	Whole numbers	Fractions	Rational numbers	Real numbers	Complex numbers
There is no number between 0 and 1.	true	false	false	false	false
Zero is the smallest number.	true	true	false	false	false
There is no solution to the equation x^2-2=0.	true	true	true	false	false
There is no solution to the equation x^2+1=0.	true	true	true	true	false

Many students are unaware of these and other differences between number systems, mainly because these systems are taught in different grades by different teachers. But also many notice that that at least some rules change when they move from one classroom to another. Inconsistencies in the material presented do not influence students' use of algorithms, but show up when students are required to reason.

The second source of contradictions is a lack of distinction between *definitions* of mathematical concepts and their *representations*.
For example,
One can represent the whole number 6, by a set of six objects, a line six inches long, a 3 by 2 grid of squares, Cuisenaire rod, and in many other ways.
But if you say that a whole number IS the cardinality of a finite set, and later you say that it IS a point on a number line, these two statements are contradictory because points on a line are not cardinalities of any sets. Here is an example from the Core Standards.

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page 43 from the Core Standards:
"Apply and extend previous understandings of numbers to the system of rational numbers.
....
Understand a rational number as a point on the number line.
....
Understand the absolute value of a rational number as its distance from 0 on the number line;
..............."
Just a moment! If a rational number is a point, and its absolute value is a distance, then the absolute value of a rational number is not a rational number, because distance is not a point.

Some contradictions come from inappropriate definitions. One example is calling all arguments of multiplication "factors" (if x*y = z, then x and y are factors of z).
It creates problem for integers, and later for polynomials, where factorization means a product of elements which do not have reciprocals. The related concept of a prime number is usually defined as a number that has exactly two factors, which is correct only for whole numbers, and does not cover prime elements in the ring of integers and rings of polynomials that are studied in higher grades.

Finally, misconceptions are created not just by providing false information. Students form many strong opinions outside school about issues important in mathematics, such as logical arguments or the structure of space. If they are not shown the difference between their beliefs and how these concepts are used in mathematics, then mathematics doesn't make sense for them.

For example, when students were asked whether the sentence, "If 0 = 1 then 2 = 6" is true, false, or neither true or false, only very few students chose the answer "true", which corresponds to the use of implication in mathematics.
When students were asked whether a straight line *is filled with points* or *consists of points with empty space in between*, almost half chose the second answer.

The word "proof" occurs five times in the Core Standards (four times in the context of geometry). But the problem of the knowledge that students bring in from outside the classrooms is never addressed.

3. Confusions created by contradictions

In order to check how much confusion inconsistent material creates, it is enough to ask questions that combine topics taught in different grades. Here are some examples.

A number is even if it has 2 as a factor. Now we know that 2*1.5 = 3.
Does it mean that 3 is now even? What do you think?

Two examples of the answer given by high school graduates.
"Three is even in math classes, except when you discuss fractions. Then it is two times one and one half. In science classes every number can be divided by 2, and 3 = 2*1.5"
"You make 3 even and then you can divide it by 2."

Is -3 a prime number? It has factors 1, -1, 3, and -3, because -3 = -1*3, and -3 = 1*-3. What do you think?
It can be followed by the next question:
What about 3? It also has factors 1, -1, 3, and -3. Is it still prime?

Is i > -i?

Is 2 a factor of $2x + 2$? Is 2 a factor of $2x + 1$? $2*(x +.5) = 2x + 1$.

And our favorite for high school students: Can you factor $x^4 + 1$?

There is a good reason why teachers avoid such questions. You don't ask your students questions to which you don't know the answers.

4. Ways to avoid confusions

Many contradictions disappear if teachers are careful that *all* statements about numbers are true in the system of real numbers.
For example, first graders can be told that zero is the smallest *whole* number, but there are other numbers some of which are less than zero, which are discussed in higher grades.
Such changes are rather easy to implement by introducing them first in college math courses for elementary teachers.
Incorrect or inappropriate definitions have to be discussed one at a time, so we'll not talk about them here.

Finally, there is no easy way to avoid contradictions between material taught and knowledge brought by students from outside the classroom. But this topic is clearly outside a discussion of the Standards.

5. Conclusion

The problems we mentioned are not limited to the United States, but they are probably less harmful when teaching is oriented more toward skills than toward understanding (which can hardly be recommended), or when more effort is directed toward making school mathematics mathematically correct.

But we see the Common Core State Standards as a missed opportunity to improve math education in the United States.

Reference

Common Core Standards in Mathematics, http://www.corestandards.org/the-standards/mathematics