Two Models of Learning the Concept of Whole Numbers

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Patricia Baggett • Department of Math. Sci. • New Mexico State University • Las Cruces, NM 88003-8001 • baggett@nmsu.edu
Andrzej Ehrenfeucht • Computer Science Department • University of Colorado • Boulder, CO 80309-0430 • andrzej.ehrenfeucht@colorado.edu • http://www.math.nmsu.edu/~breakingaway/

1. INTRODUCTION

Technology has made most written arithmetic computations obsolete. So the goal of teaching arithmetic in elementary and middle school has switched from teaching arithmetic skills to making children ***understand arithmetic***. But the content of what is taught has changed very little and is still based on verbal counting and memorization, which are the background skills for written algorithms. Yet learning these skills does not provide a good background for understanding arithmetic.

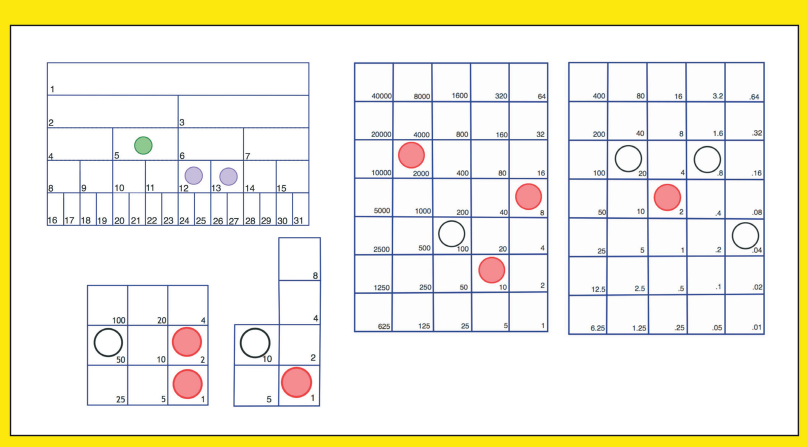
We propose a different approach to teaching arithmetic, following the approach of ***John Leslie***, who in his *Philosophy of Arithmetic (*1817) based arithmetic not on counting, but on ***halving***, which is needed for partitioning a quantity into two equal parts.

Arithmetic algorithms based on the idea of halving were developed much earlier by ***John Napier*** and described in his *Rabdology* (1617) under the headline of "location numbers".

The ***algorithms*** proposed by Napier are ***easy to learn, flexible,*** and ***efficient***.

The works of Leslie and Napier are still mostly ignored, in spite of an article by ***Martin Gardner, Napier's Abacus,*** in *Knotted Doughnuts* (1986).

We don't propose any drastic changes in current teaching practices. But we suggest that children be shown how to plan and carry out arithmetic computations on one or more counting boards which we have designed, based on the work of Napier, Leslie, and some modern concepts of computer science.



2. HISTORICAL BACKGROUND

1. The model that is most often used was designed by Guiseppe Peano in *Arithmetices Principia: Nova Metodo Exposita* (1889). Peano uses n+1 (a successor operation) as a basic arithmetic operation to create a sequence, 1, 2, 3, …, (which is now often called "the ***number line***"). Next he provides a recursive definition for adding two numbers in terms of n+1, and a recursive definition for multiplication in terms of addition. (All other arithmetic operations can be defined from addition and multiplication by explicit definitions.)

This system of numbers (called "***natural numbers***"), is extended by adding to it "ratios" of natural numbers (fractions), their "opposites" (negative numbers), and the number zero. Extending the operations of addition and subtraction (and their inverses) completes the construction of rational numbers.

2. John Leslie thought that the invention of numbers was motivated by a need for a fair partition of a collection of objects into two parts, as equal as possible. It can be done by the operation of halving (one for me, one for you) with a remainder one. Then Leslie constructs natural numbers using two basic operations, n+1 (***successor***) and 2*n (***doubling***). These two operations together form a structure that is now called a ***binary heap***, or binary tree.

Heap of natural numbers up to 31

1
2 3
4 5 6 7
8 9 10 11 12 13 14 15
16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31

The numbers on a heap, when scanned line by line, form an arithmetic progression of a number line. But all columns form geometric progressions with the quotient 2. And this feature provides "fast access" to each number in a heap. All definitions used by Peano can also be used in Leslie's model. But many other definitions, which use the concept of doubling, provide a basis for more efficient methods of computation than the original definitions of Peano.

The ***heap boards*** are counting boards on which natural numbers are arranged in a ***binary heap***. Heap boards are not useful in practical computations because the range of numbers represented on them is very small and limited only to integers.

3. PRACTICAL COUNTING BOARDS

John Napier's "abacus" was a rectangular counting board on which values of locations formed geometric progressions with quotient 2, both in columns and in rows.

…	…	…	…	…	…
…	**64**	**32**	**16**	**8**	**4**
…	**32**	**16**	**8**	**4**	**2**
…	**16**	**8**	**4**	**2**	**1**

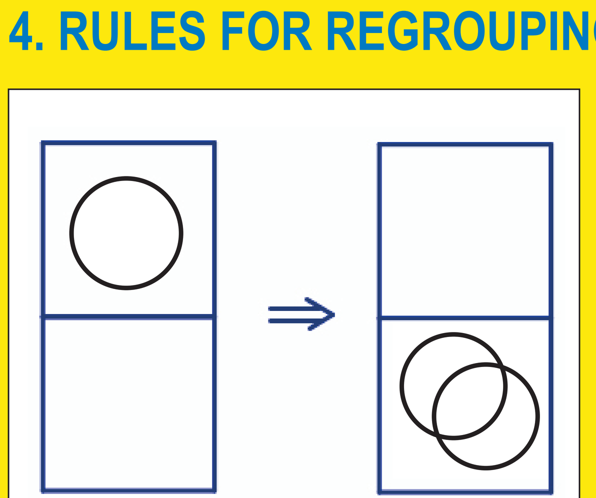
He showed that the four arithmetic operations, addition, subtraction, multiplication, and division, and computations of roots are carried out very efficiently on his board. This is not surprising, because the underlying principle makes his board similar to a two-dimensional slide rule, and not to other counting boards that were used in Asia and Europe. Unfortunately he represented the numbers in his own invented binary notation, which was probably the reason his invention was ignored.

Decimal boards are a modification of Napier's invention, where rows hold geometric progressions with quotient 5 rather than 2, and are extended to fractional values. And columns are also extended to fractional values.

…	…	…	…	…	…
…	**50**	**10**	**2**	**.4**	…
…	**25**	**5**	**1**	**.2**	…
…	**12.5**	**2.5**	**.5**	**.1**	…
…	…	…	…	…	…

Different ***tokens*** put on a board can have different values. We use two-color tokens. White always has the value 1, and red has the value -1. But other colors or markings on tokens can indicate other values. The value of a token on a board is always the product of the value of the token and the value of the location. The total value of a board is the sum of the values of all tokens. Some computations may require the use of several boards used at the same time.

4. RULES FOR REGROUPING TOKENS



A rule for regrouping tokens describes how to change a pattern of tokens on a counting board without changing the total value on the board. We also require that a rule ***can be used everywhere*** on a board. For each type of board, heap and decimal, there is a simple set of rules that are sufficient to carry out any computation. An example of one rule on any decimal board: You may replace one white token by two white tokens at a location just below.

Implementing any arithmetic algorithm on a board looks as follows: You "put a number" on a board, and regroup. You add more tokens and regroup. You continue until you get a solution, which you "read" from the board. The numbers you put on a board vary, depending on the arithmetic operation whose value you are trying to compute, and the specific algorithm you use. But ***the rules of regrouping remain the same.*** They allow you to simplify the patterns of tokens and arrange them in a way that is easy to read as a decimal. Because the rules of regrouping remain the same, no algorithm is a prerequisite for any other algorithm.

5. PARTICIPANTS, PROCEDURE, AND DATA COLLECTION

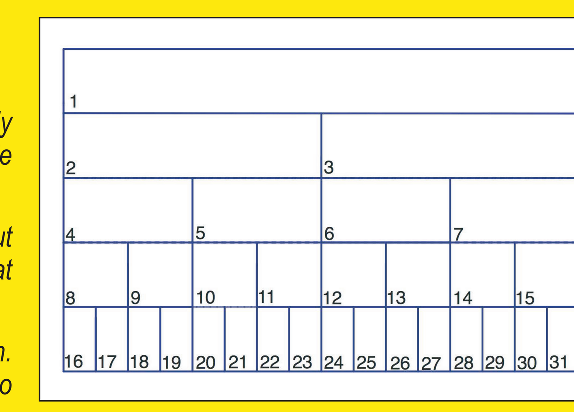
Twenty-nine students from an undergraduate statistics class for future teachers, and 12 undergraduate and graduate students in an algebra and geometry class for current and future teachers were shown how to use both decimal and heap boards for selected arithmetic tasks during the first fifteen sessions of their courses. During the sessions the instructor led them in learning an operation on the board, and then they practiced one or more similar problems. Each session lasted about 15 minutes, for a total time of about 4 hours.

The topics for the decimal board covered the basic rules of regrouping, adding positive and negative numbers, one algorithm for division with remainder, two algorithms for multiplication of decimals, and some other tasks such as, "In how many ways can you represent the number 3 with two tokens on a decimal board?" ***(The answer is 8, but finding them all is not easy!)***

The topics for the heap board, which is simpler to use, but which was introduced after the decimal board, included addition of integers, and changing a binary representation of a whole number created by "counting by halving" into a base-10 number. Students' work in class was graded on the basis of bi-weekly writing assignments. But because they always had some choice among different topics, only some students wrote about the counting boards.

SOME COMMENTS IN THEIR WRITINGS

Comments by students about the heap board



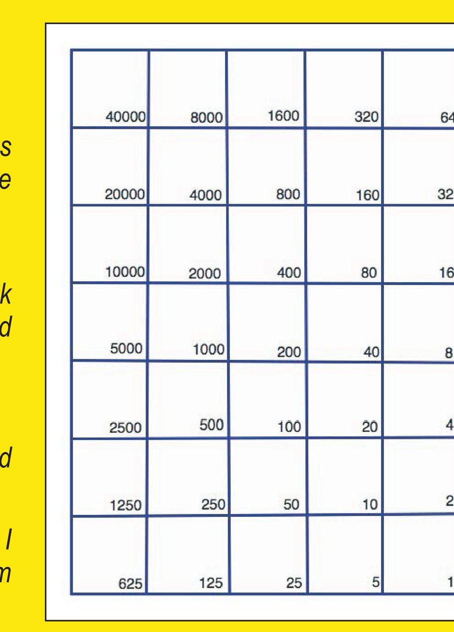
- We were introduced to a new board that is numbered from 1-31 and is intended for early childhood. I really like the board because it helps children not only add and subtract, but to see the numbers on the board while they work the problems.

- This new and improved counting board for young learners is amazing! Students can not only learn about evens and odds, but they can also learn about addition and subtraction. And students can visually see what the change is and what is happening.

- This board shows a fun way of adding and subtracting instead of just doing it with pencil with an equation. Using different colored tokens, students can keep track of where they are in the problem they are trying to solve.

- I would definitely use the "heap" board to teach adding and subtracting to younger grades because it gives a visual learning experience for the students. They are not just looking at numbers on a page (e.g., 12 – 3 = ?). It allows the students to see the little tokens move.

Comments by students about the decimal boards



- I think the boards can help students understand how addition, subtraction, multiplication, and division actually work.

- Using the boards gives students a different way of solving complicated multi-step problems. It also treats math as more of a game and a puzzle challenge, which engages their interest and keeps them motivated. This may help some students that struggle with traditional methods to be able to grasp the concepts better.

- This activity was very fun to do! I enjoyed learning to use the counting board to add, subtract, multiply, and divide. Addition and subtraction were easier to do than the other two. I think this activity is great because children start to think about numbers in a whole different way. I tried this with my 7-year old daughter… and I like that her brain is working and coming up with different ways to make a certain amount…

- It feels like you're playing a game instead of making equations.

- Learning division on the counting board will teach children that doing math is a lot of fun and should not be so hard and it helps them understand the process.

- I really enjoyed learning how to do division on the counting boards. The whole concept really made sense to me, and I think it would also make sense to students. I believe if you start teaching students how to use the counting boards from a young age, they will gain a more in-depth understanding of math operations.

QUESTIONNAIRES AND RESULTS

Questionnaire for statistics class

We have shown you two kinds of counting boards. We really want to know your opinions about how useful they can be in teaching math in schools. So take your time to answer the four questions, two below and two on the back of this sheet.

DECIMAL BOARDS
1. Are you familiar enough with decimal boards to show them to somebody else? Yes Possibly No
2. Do you think that decimal boards would be useful in teaching math in schools? Yes No opinion No

Explain your opinion by talking about different level of students, preschool-kindergarten, elementary, middle school, and high school. Also talk about different tasks for which the decimal board may be used. *(Use the back of the page to continue.)*

HEAP BOARDS
3. Are you familiar enough with heap boards to show them to somebody else? Yes Possibly No
4. Do you think that heap boards would be useful in teaching math in schools? Yes No opinion No

Explain your opinion as above, and also you may compare the use of both kinds of boards.
(You may use the space below, or you may ask for an extra sheet of paper.)

Results for statistics class

Answers to the question whether they could explain the boards

	Decimal	Heap
Yes	6	17
Probably	15	6
No	4	1

Answer to the question whether a board should be used in schools

	Decimal	Heap
Yes	18	23
No opinion	6	1
No	1	0

Questionnaire for algebra and geometry class

Questionnaire about the decimal counting board

1. Which activities that we did in class do you still remember? *(If something was done in more than one way, you can count it as more than one activity.)* Mark each one that you remember well enough to show to someone else.
2. We want to know your opinion:
Do you think that the board should be used as a teaching aid in schools? If yes, in which grades?
We are especially interested in how you would justify that it should be used or that it should not be used. Please tell us below, and use as much space as you want (including the back) to justify your answer.

Second Questionnaire for algebra and geometry class

Last week you were shown a heap board, which has a different design from the decimal boards. Here are a few questions:
1. Which of the two types of boards, heap or decimal, is more interesting for you?
2. If both types of boards were used, which type is suitable for which grades?
3. Which boards are more suitable for what tasks? *(Here you don't need to justify your answers; just say what you think.)*

Results for algebra and geometry class

Which board is more interesting?

Equally interesting:	4
Decimal:	5
Heap:	2
Total:	11

SUMMARY OF OPINIONS OF 9 (OF 11) STUDENTS ON QUESTIONNAIRES ABOUT THE USE OF THE BOARD

Heap boards should be used in early grades to teach counting, addition and subtraction.

Decimal boards should be used in later grades to teach all arithmetic operations.

For further information, please contact Pat Baggett, baggett@nmsu.edu.