A mathematically rigorous calculus course in a laboratory format for undergraduate and graduate non-math majors

Joint Mathematics Meetings

MAA Session on Innovative Approaches to One-Semester Calculus Courses

Seattle, WA

January 7, 2016

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    Box and ladder

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1. Introduction

We offer a one-semester calculus course at New Mexico State University with the following properties:

- The course may be taken for graduate or undergraduate credit. Most students are prospective or practicing teachers.
- There are no prerequisites.
- The course is run in a lab format (most work is done in class).
- Students work individually or in small groups.
- Calculator and computer technology are used.
- Each topic is introduced through a specific task. And most tasks have a “hands-on” component.

2. Three tasks

a. String around a can

Task:

Tie a string around the vase to form a coil of a spiral. Measure the height and diameter of the vase. Then,

(a) write parametric equations that describe the spiral in 3D, and compute its length using an integral formula;

(b) compute its length using the Pythagorean Theorem; and

(c) measure the length of the string that forms the spiral.

• Here is a mathematical model of the situation which is the formula for the length of the string.

t is an angle measured in degrees around the axis of symmetry of the can.

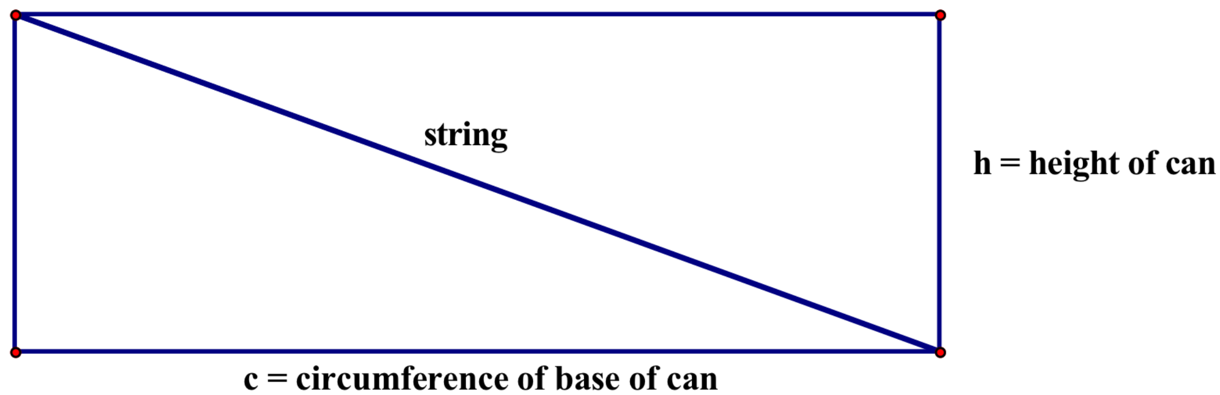
x = D/2*cos(t) one horizontal coordinate;
y = D/2*sin(t) another horizontal coordinate;
z = (H/360)*t vertical coordinate

When the string goes around exactly once, the arc length is

$$\text{length} = \int_0^{360} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

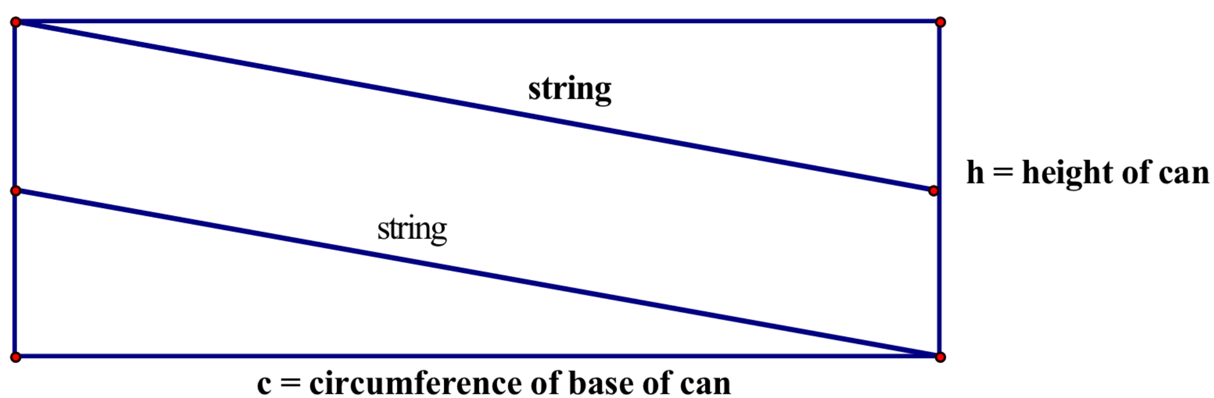
This integral can be computed using fnInt and nDeriv on the TI-83/84.

- Here is a very elementary and obvious way to solve the problem using the Pythagorean theorem:



string goes once around can; string length = $\sqrt{c^2 + h^2}$

or, if the string goes around twice,



string goes twice around can; length = 2* $\sqrt{c^2 + (.5h)^2}$

Two important questions that need to be addressed while solving any problem:

(1) Are we using a correct method?

(2) Is the answer correct?

Writing the parametric equations

x=D/2*cos(t)	one horizontal coordinate;
y=D/2*sin(t)	another horizontal coordinate;
z=(H/360)*t	vertical coordinate

and computing the length of the curve using

$$\text{length} = \int_0^{360} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

were new and difficult for students.

So showing how to do it with the Pythagorean Theorem gave intuitive validation for the general method.

And finally, measuring the length of the string showed that the answer was correct.

The informal definitions of *derivative* and *integral* that were used in this course were

derivative: “Rate of change of one variable relative to the change of another variable”
And

Integral: $\int_a^b f(x)dx$

“The average of the values of the function f, where x is between a and b, times b – a.”

The more common explanations of these concepts,

derivative as “the slope of a tangent line”,
and

integral as “the area below the graph of the function”

were not used.

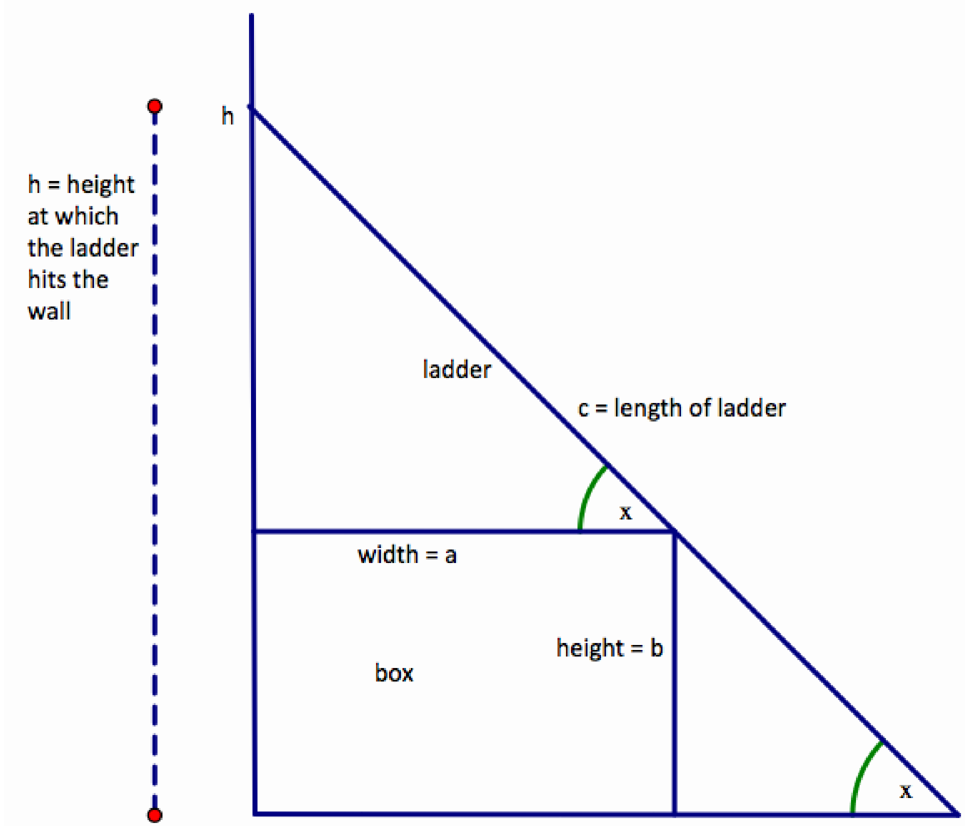
In this task these explanations can be confusing.
The integral was used to compute a length and not an area.
And the derivative was not the slope of a tangent line.

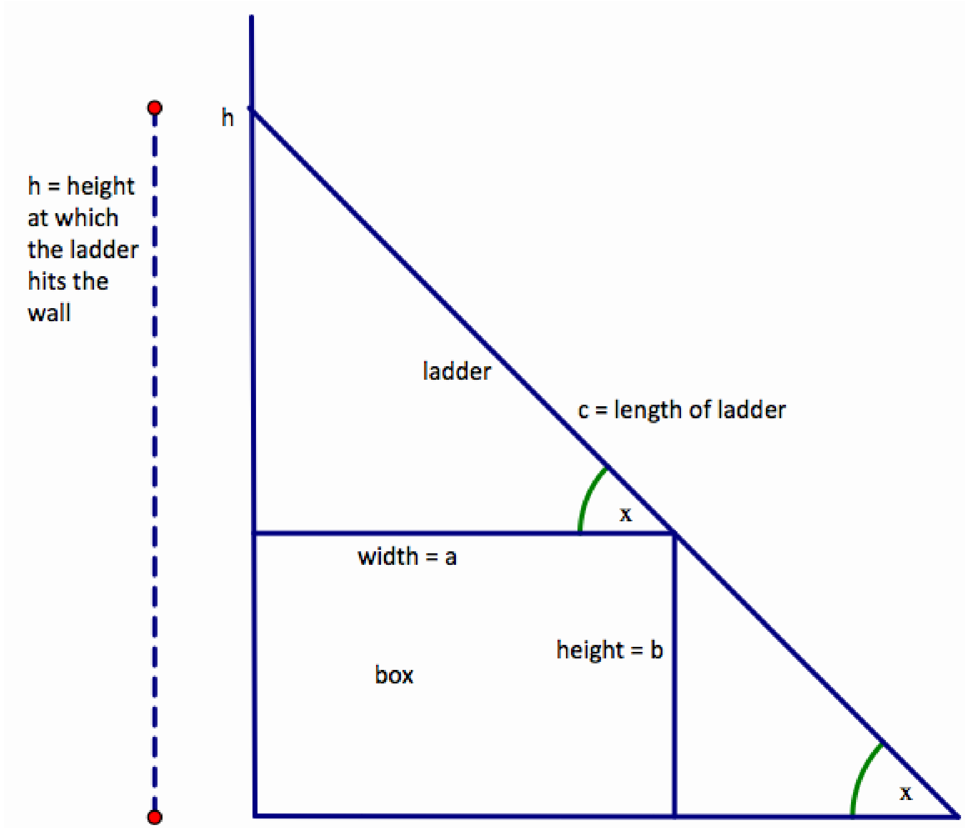
b. Box and ladder

Task:
A box with width a and height b stands next to a wall. You put a ladder of length c next to the wall over the box as shown in the picture.
At what height h does the ladder touch the wall?

This is a well-known problem.

In the case of a cubic box, its abstract version was solved by Newton in his *Arithmetica Universalis* (1707, 1720).


h = height
at which
the ladder
hits the
wall
h
ladder
c = length of ladder
x
width = a
box
height = b
x



The general solution for a box whose sides have lengths a and b, and whose ladder has length c, can be obtained by solving for x the equation:

$$b/\sin(x) + a/\cos(x) = c$$

And then computing the height $h = c*\sin(x)$.

In the class, most of the time was spent on deriving the equation, and on the required knowledge of geometry and trigonometry that are “prerequisites” for this problem.

The equation, which doesn’t have an elementary solution, was solved with the calculator.

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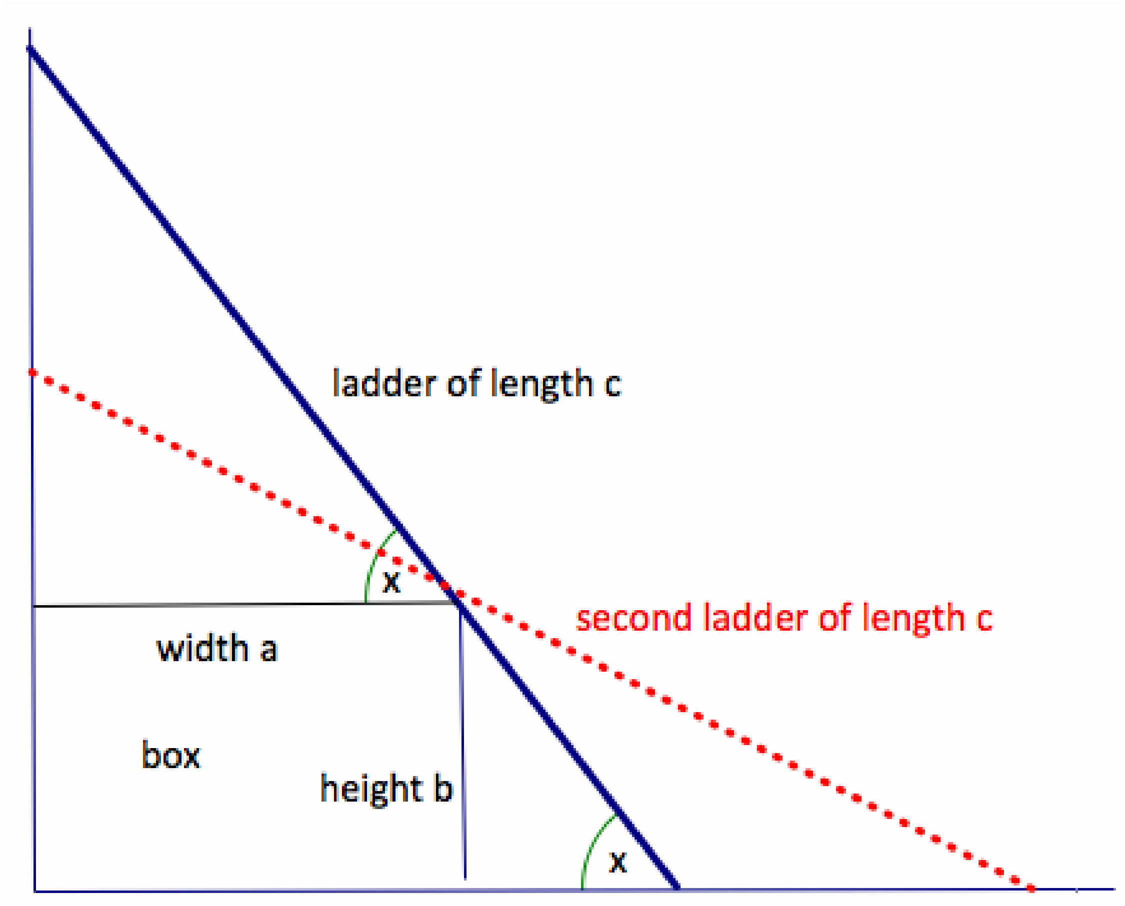
On the TI-83/84 calculator, equations are solved by Newton's method. But this topic was not included in the task.

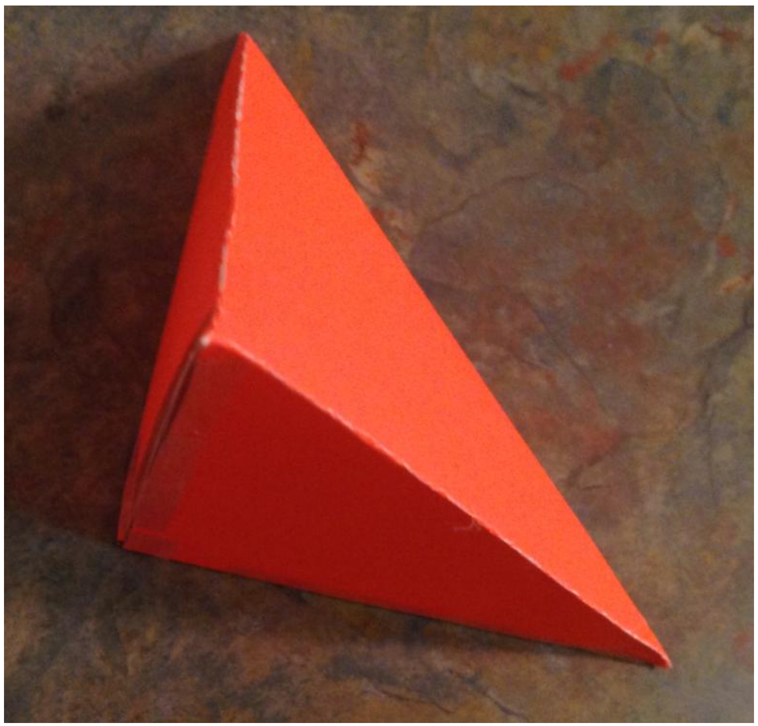


In almost all cases the problem has exactly two solutions.

Students had "ladders" (sticks) and boxes. They measured how high their "ladders" reached the wall, and how far from the wall the ladder touched the floor, to see how well the theoretically derived predictions matched the actual measurements.

The fit was almost perfect!

c. Finding the volume of a “right” tetrahedron



Definition: A right tetrahedron is an irregular tetrahedron with 3 faces that are right triangles with legs of length h, and with one face an equilateral triangle with sides √(2)*h.

Task:
Make a net for such a tetrahedron with h=10 cm, and build it from poster board. (Check that 8 of these make a regular octahedron.)

Compute the volume of the tetrahedron using four different methods:

1. using three different methods of computing integrals (discussed in the class)
2. using the formula for the volume of a pyramid

In the course the indefinite integral was introduced by making the upper limit of a definite integral a variable:

$$\int_a^x f(t)dt$$

where x is a variable.

This corresponds to the implementation of the indefinite integral on the TI-83/84, fnInt.
But the method that is implemented (the Gauss-Seidel algorithm), cannot be explained in an introductory calculus class.

So in this unit students were shown two other methods of computing approximate values for both definite and indefinite integrals:

So in this unit students were shown two other methods of computing approximate values for both definite and indefinite integrals:

One is based on an intuitive explanation of an integral.

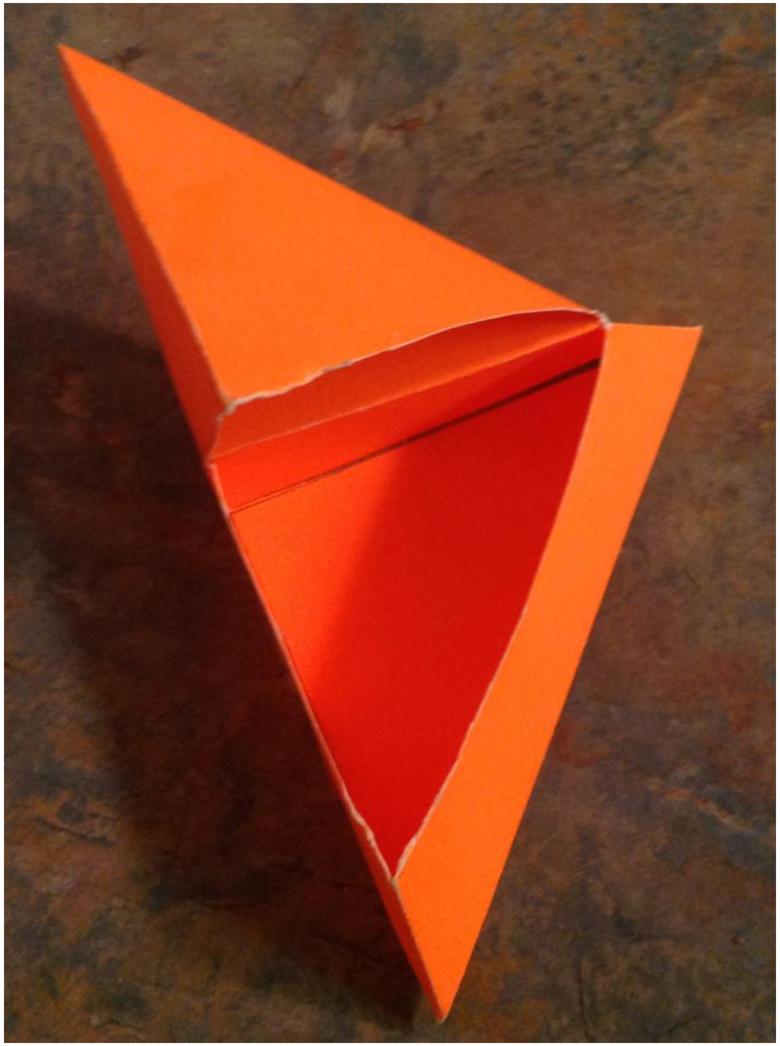
Take the mean of a random sample from a uniform distribution of values of t from an interval [a, x], and multiply it by x – a.
This is an approximation that can be justified by Lebesgue's definition of integral, but not by Riemann's definition.

The other (similar to Simpson’s rule) can be computed directly from a representation of a function f(t) that consists of two lists:

a list of values of t, and
a list of matching values of f(t).

This method provides results with high accuracy with lists that can be easily defined on scientific calculators.

Computing an indefinite integral by this method is also much faster than using the corresponding calculator function fnInt.



All three methods use slices, i.e., right-triangle cross sections.

But the number of slices, their location, and how they are used in computation are different in all of them.

Representing functions by pairs of lists is a very useful technique for computing derivatives, integrals, and other operations on functions.

But it is much too cumbersome for hand calculations. It became practical only after the invention of computers and scientific calculators.

The exact formula for the volume of a pyramid,

$$\frac{1}{3} * \text{area of base} * \text{height},$$

provided an easy comparison for the accuracy of all three methods of computing an integral that the students used.

3. Students’ opinions about the course

The course has been taught six times.

Each time the selection of topics was somewhat different, and each time different aspects of problem solving were stressed.

In fall 2015 we focused more on the numerical aspects of computation, and on calculator programs that were used. We gave students the following questionnaire:

In this course we used:

1. hands-on activities
2. intuitive meanings of mathematical concepts
3. We did not use mathematical definitions that require the concept of limit.
4. Instead, we showed how to carry out computations on TI-83/84 calculators.

Which of these four features were good aspects of the course? Which were bad? And which were in between?

Some comments

I love incorporating the TI-83/84 calculator…

I have learned so many cool things about my calculator. But *I did not always understand what it meant*…

…All four features contributed positively to the course overall…. But I would have liked to see what it would be like *to use a mathematical definition to solve something* in comparison to the way we worked without them….it probably wouldn't be as much fun!

I feel like each feature was integral (math joke!) to the course being effective and purposeful and I found each aspect to be exciting and informative in different ways. I think I only *struggled with some of the newer and more elaborate calculations and inputs* due to my being a novice and at times being a bit rushed.

4. Final remarks

Teachers and future teachers who take a calculus class want to learn material in a way that enables them to teach it in their classrooms. So all details that may be relevant for solving each problem that is studied have to be addressed.

Providing detailed lesson plans that clearly state tasks, methods of solution, and an explicit criterion for checking whether the answer is correct, is very helpful.

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Criteria for checking whether the solutions are correct are especially important in calculus courses, because the methods used to solve problems are not obvious and are unfamiliar to beginners.

All the relevant details for a problem need to be provided, and this strongly limits the scope of the material that can be covered in a one-semester course.

But creating lesson plans that are self-contained and do not require prerequisites allows us to teach advanced topics even in one semester course.

For lesson plans used in the course, including those on computing the numerical integral (and the numerical derivative) with lists on the TI-83/84, see

https://www.math.nmsu.edu/~breakingaway/ebookofcalculus/

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Thank you!