A Prehistory of Arithmetic

History and Philosophy of Mathematics
MathFest
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Outline

1. A need for a revision of the prehistory of arithmetic

2. John Leslie's hypothesis (1815)

3. John Napier's contribution (1617)

4. Arithmetic taught in schools

5. Interesting mathematical problems

1. The need for a revision of the prehistory of arithmetic

The picture of arithmetic in preliterate societies drawn in the 19th and 20th centuries was based on comparative linguistics, ethnography, and very limited archaeological evidence.

The evolutionary view of “progress” viewed early humans as savage brutes who used clubs and spoke in monosyllables.

So it was assumed that arithmetic could not have been abstract, and that it was limited to making tallies, counting, and not much else.

At present it is assumed that at least since the time of Cro-Magnons (~40,000 years ago), early humans were intellectually equal to modern humans.

So the question is not *whether they could have* developed an abstract arithmetic, *but whether they did*; and *how it could have been* done.

2. John Leslie’s hypothesis

In his book *Philosophy of Arithmetic*, published in 1815, John Leslie proposed the hypothesis that the invention of arithmetic was not motivated by the need for *counting*, but by the need for *sharing*.

The simplest fair sharing procedure is “*one for me and one for you*”.
It doesn’t require that participants know how to count, or that they have number words for more than 3.

Then Leslie says that the beginning of arithmetic stemmed from observing that any collection of objects is either “*even*”, i.e., it can be split equally, or “*odd*”, it can be partitioned into two equal parts with a remainder of one.

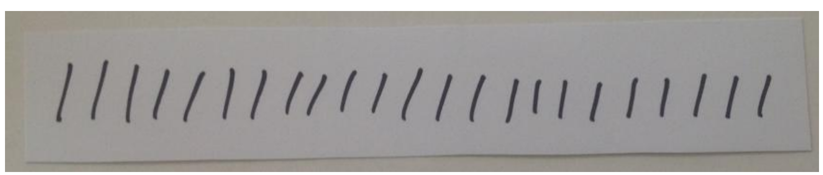
The key discovery was that, when one repeats taking halves of a collection of objects, creating halves, halves of halves, and so on, and one records whether the amount is “even” or “odd” each time, the resulting “even-odd” record *uniquely describes the original amount*.

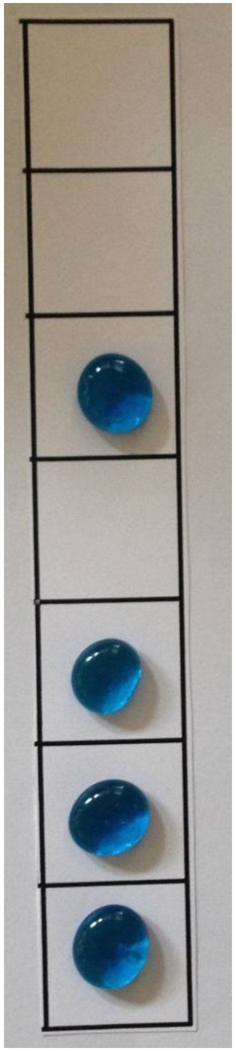
The construction of *even-odd records* can be viewed as *the beginning of arithmetic*, because it allows one to *simulate processes on physical quantities* by *processing records representing those quantities*.

So, for example, suppose we have this collection of stones:

It can be represented by this record:

It can be represented by this tally:

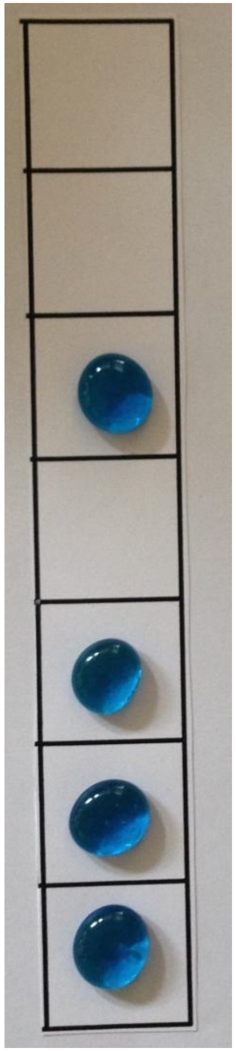


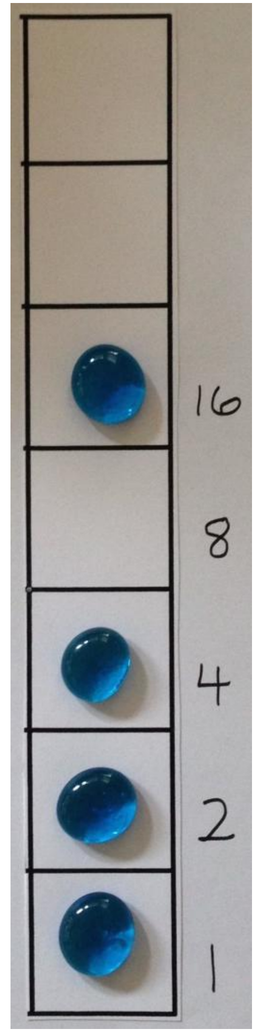


On the previous slide you saw 23 stones, a tally of 23 marks, and a record of the result of repeated halving: 23 odd, 11 odd, 5 odd, 2 even, 1 odd.

Odd results are represented by blue tokens listed from bottom to top. Empty squares represent even results. The top token indicates the end of a record.

The record can be read top down as 1 0 1 1 1, where “odd’ is 1, and “even” is 0.
This is 23 written in base two.



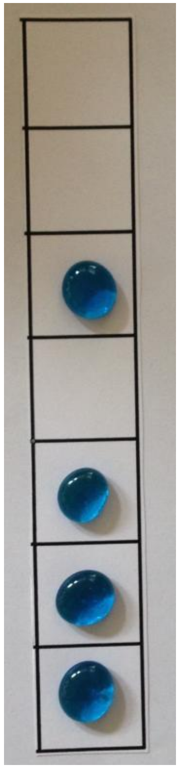


10111
oeooo

Thus, from the very beginning, arithmetic operations *could have been operations on records,* and not on quantities represented by records.

When we start with “halving” (with remainder) as a basic arithmetic operation, we have two inverse operations to construct bigger numbers, “doubling” and “adding one”.

For example, a record:



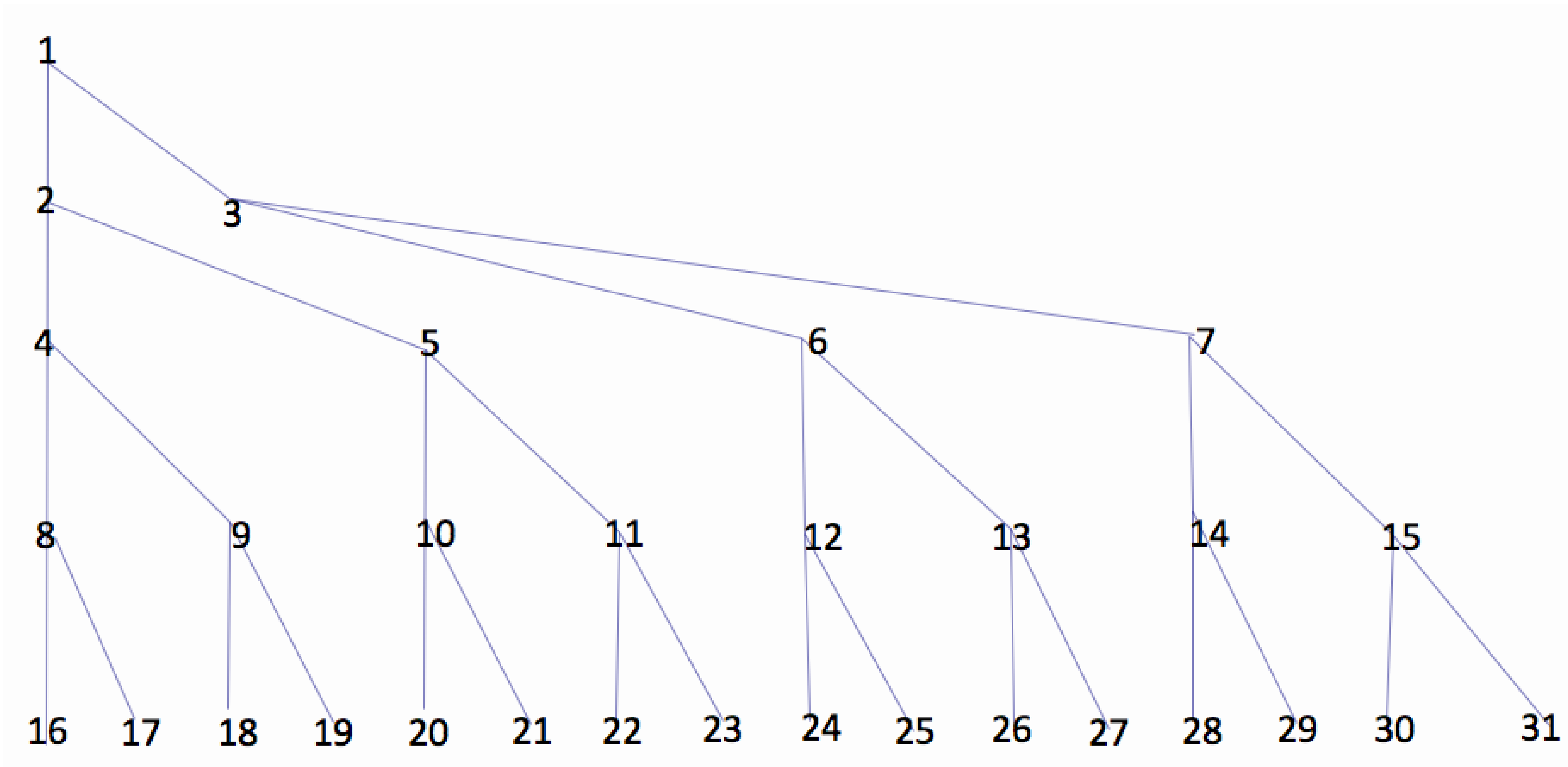
Its double:

And one more:

In modern jargon we can say that Leslie assumed that the first arithmetic structure was not the *number line*, as most other authors have assumed, but a *binary heap*, which, when read horizontally, gives a number line; but it also has a vertical structure of a “binary tree”.

One way to draw a binary heap of the first 31 positive integers:

1															
2								3							
4				5				6				7			
8		9		10		11		12		13		14		15	
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31



The same heap as a tree, showing *double* and *double plus one*

Remark

Even-odd notation is equivalent to modern base 2 notation.

But base 2 is defined in terms of three binary operations, addition, multiplication, and exponentiation,
whereas even-odd notation and its corresponding binary heap require only two unary operations, double and add-one.

Leslie's hypothesis has another equally important aspect.

There is no reason to assume that sharing was restricted to discrete collections of objects.

Sharing other quantities could have been even more important.

And this suggests that the development of the concept of a fraction as one of the equal parts of a whole (an *aliquot part*) could have been part of arithmetic from the beginning, rather than being introduced later and defined in terms of whole numbers.

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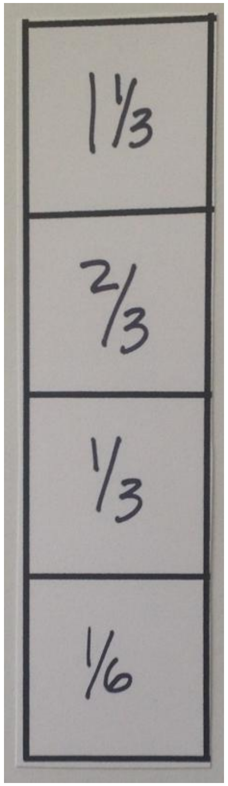
But in order to make Leslie’s hypothesis believable, we need to show that preliterate groups of people could have developed tools for doing arithmetic that were powerful and flexible enough to compete against written methods developed later by literate societies.

3. John Napier's contribution

In the last chapter of *Rabdology* (1617) John Napier described a very flexible and ingenious way of representing numbers on a "chessboard" which allows one to carry out the four basic arithmetic operations (and much more) at least as efficiently as with paper and pencil.

A basic unit is a "rod", which is a finite sequence of locations labeled by a geometric progression of numbers. The quotient of the progression is two, on all rods. A board is a set of rods arranged in a pattern convenient for a user.



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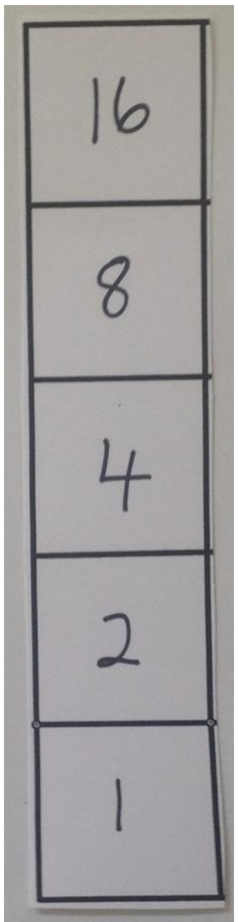

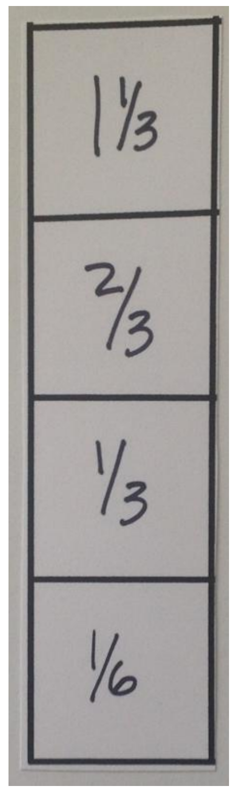

A token put on a board acquires the value of its location.

Examples of some small boards
(In these examples rods are shown as columns of numbers.)

1000	200	40	8
500	100	20	4
250	50	10	2
125	25	5	1

120	40	24	8
60	20	12	4
30	10	6	2
15	5	3	1

2048	256	32	4
1024	128	16	2
512	64	8	1

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Notice that the columns in a binary heap of numbers (shown before) are also rods, so a binary heap of numbers is also an example of a Napier's board.

2048	256	32	4
1024	128	16	2
512	64	8	1

Important properties of Napier’s invention

(1) A number on each rod is represented, not by *the number of tokens*, but by a *pattern of tokens*, which is very efficient.

(2) The use of geometric progressions allows one to carry out multiplication by shifting the existing patterns and adding the shifted values. The design of the slide rule was also based on the same concept of shifting a pattern of numbers. So there is a clear connection between Napier’s interest in counting boards and his work on logarithms.

(3) The rules for regrouping tokens are very simple, and a user does not need to memorize arithmetic “facts” in order to carry out arithmetic calculations. Also the rules for regrouping patterns of tokens do not use numbers assigned to locations, so number words are optional.

(4) Using different sets of rods for different calculations provides flexibility.

Computing with fractions

In the last chapter of *Rabdology* Napier doesn't talk about computing with fractions. But here is description of how it can be done.

Decimal fractions and fractions in other bases

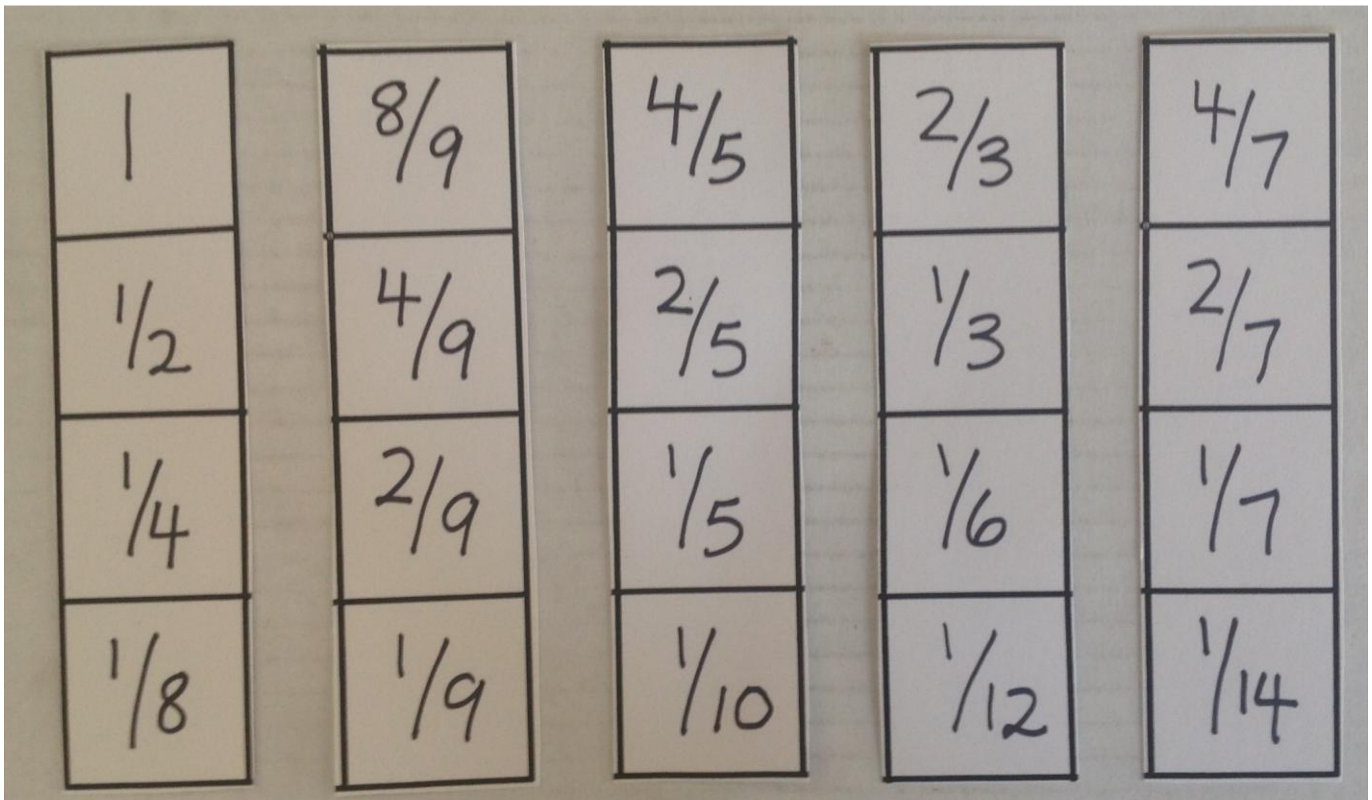
It only requires an extension of the rods "downward", so they include negative powers of two:

On this board one can carry out computation with decimals ranging from .01 to at least 2000
.

1000	200	40	8	1.6	.32
500	100	20	4	.8	.16
250	50	10	2	.4	.08
125	25	5	1	.2	.04
62.5	12.5	2.5	.5	.1	.02
31.25	6.25	1.25	.25	.05	.01

Common fractions (rational numbers between 0 and 1) require a different construction based on the reciprocals of numbers that are used to form a binary heap.

An example of 5 rods that can be used to handle fractions with denominators smaller than or equal to 10:



An important theorem, which allows efficient computation of sums and differences of fractions, states that

the unit fraction

1/lcm(n, m)

can be represented on rods containing 1/n and 1/m.

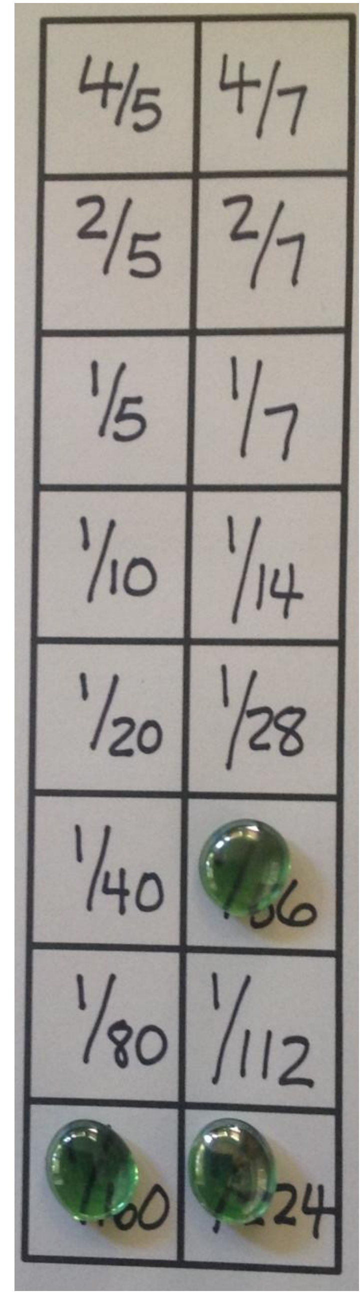
For example, 1/35 can be represented on two rods, one containing 1/5 and the other 1/7.

$$1/35 = 1/56 + 1/160 + 1/224$$

This is so, because

$$1/35 = 1/(7*8) + 1/(5*32) + 1/(7*32)$$

$$1/35 = 32/1120 = (20 + 7 + 5)/1120$$



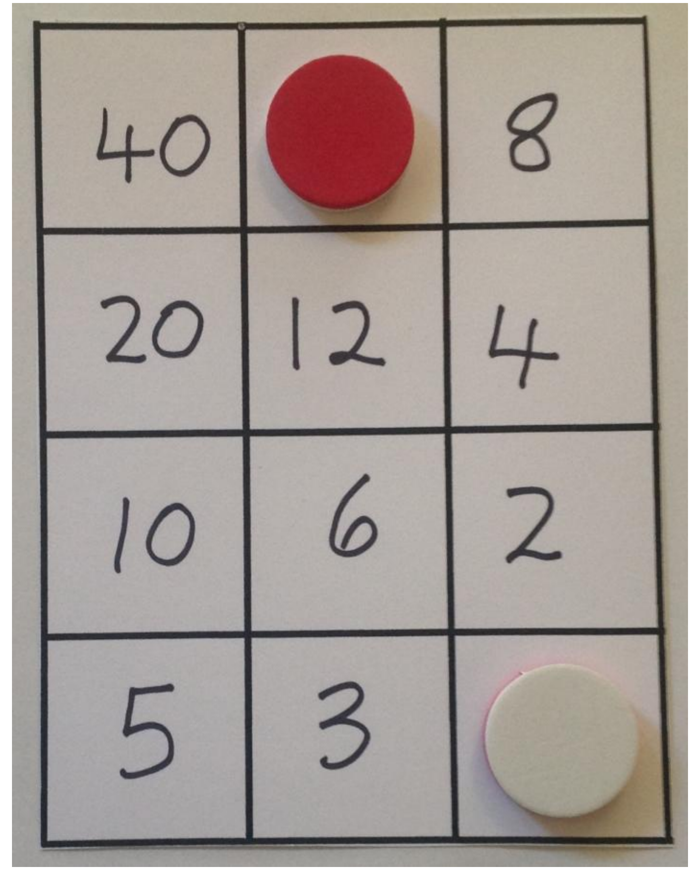
Negative numbers

John Napier doesn't mention negative numbers in *Rabdology*, but they can be represented by tokens with a value of -1.

Working with two kinds of tokens (with values 1 and -1) allows us to represent two numbers on the same board, instead of using a different board for each number.

So a negative number can be treated as "a number to be subtracted".

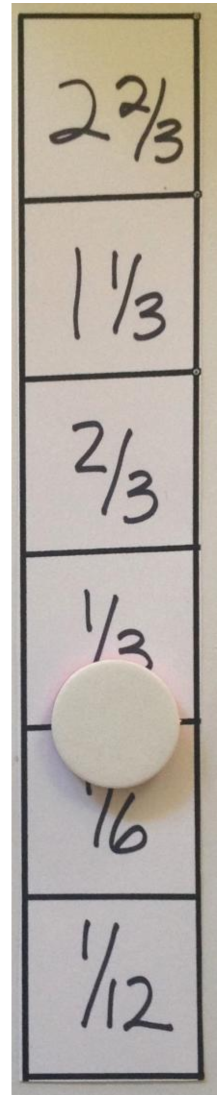
The conceptual difference between positive and negative numbers appears only during multiplication, because the product of two negative numbers is still positive.

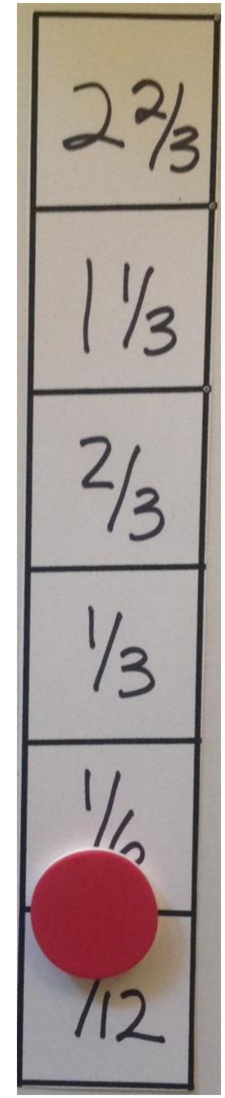


-23 is represented above.

Irrational numbers

During computation we mostly represent irrational numbers by their approximations. But there is no established method to indicate during written computation whether a number shown is exact or rounded. But putting a token on a line separating two locations having values v and v/2 on one rod can always mean that its value, x, is somewhere in between; namely, v/2 < x < v. So even when a token's value is not defined, knowing its possible range is sufficient for using it in most computations.





Above, a number between 1/6 and 1/3 is on the left, and a number between -1/12 and -1/6 is on the right.

Conclusion

The hypothesis that the beginning of arithmetic started with partitioning a discrete or a continuous quantity into two “equal” parts, and this led to the development of advanced stages of arithmetic in a preliterate society, is *theoretically possible*.

Counting boards that are designed according to the principle introduced by John Napier provide a tool for the development of arithmetic that has the same power as written computations.

It requires only the technology that was available long before the invention of writing, and it doesn’t depend on the existence of a sequence of number words.

4. Arithmetic taught in schools

At present in the United States and other industrial countries, skills in written arithmetic have very little practical value.

But they still form the core of the arithmetic that is taught in schools, even if the goal of teaching arithmetic has changed from developing skills of written computations to understanding the principles of mathematics and its use in everyday life.

And the techniques of written computation and the pedagogy developed to teach them, seem to be very poorly suited to the new goals.

So we think that we should look for a different approach to teaching elementary arithmetic in grades K to 8.

And we think that replacing written arithmetic by arithmetic on counting boards could be the right option.

Using counting boards in classrooms

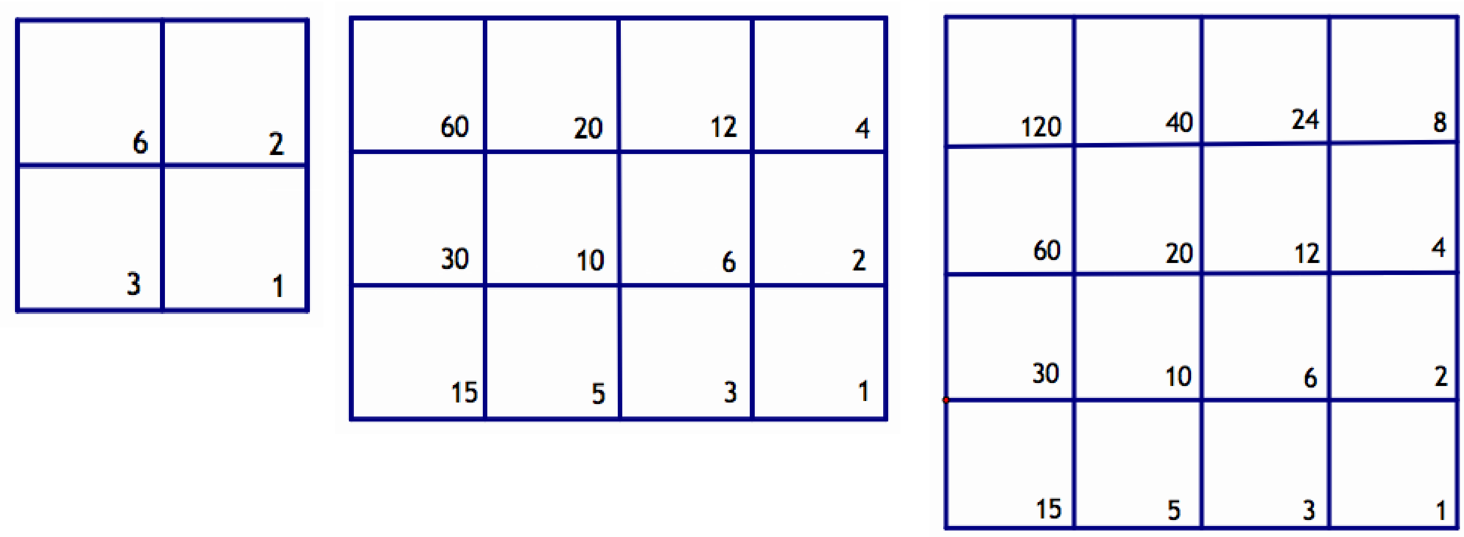
We have experimented with three kinds of boards:

* Decimal boards of different sizes and shapes where rods are arranged into an array in which each row forms a geometric progression with quotient 5.

The largest boards were 6 by 6, and some of them included decimal fractions.

100	20	4	.8	.16	.032
50	10	2	.4	.08	.016
25	5	1	.2	.04	.008
12.5	2.5	.5	.1	.02	.004
6.25	1.25	.25	.05	.01	.002
3.125	.625	.125	.025	.005	.001

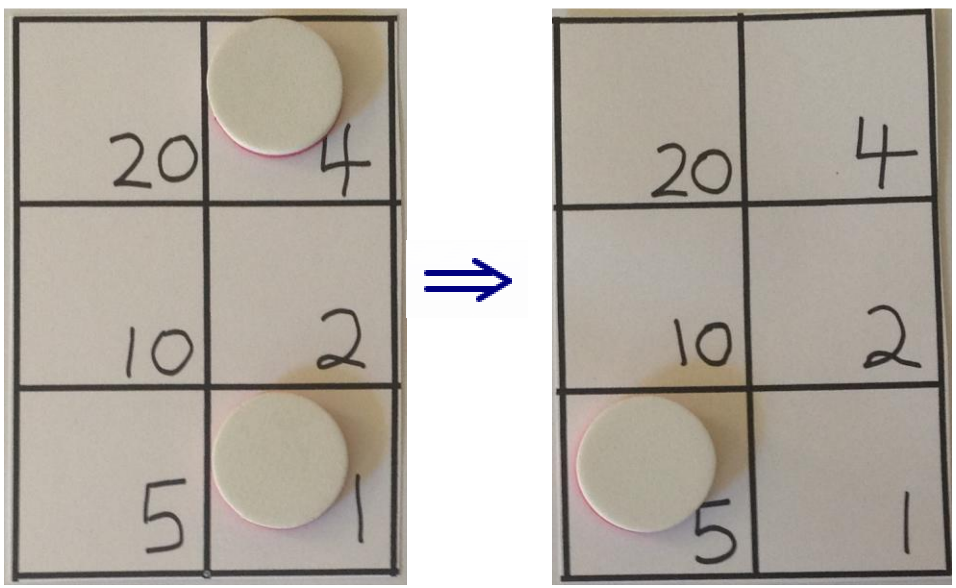
* Other small boards with different arrangements of rods:



* And “heap boards”, which we won’t discuss here.

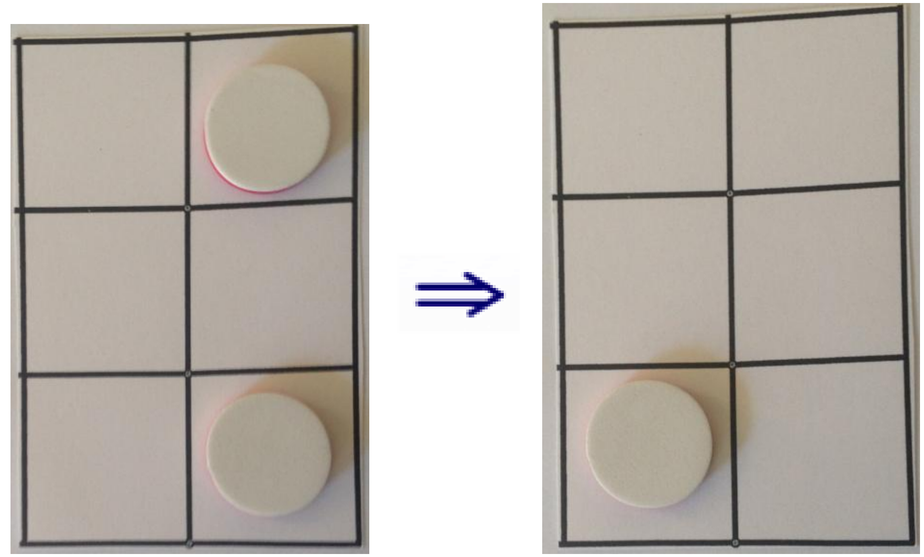
Regrouping

Each board provides a relationship between operations on numbers and moves one can make.
For example, on the decimal board above a student may regroup tokens as follows,

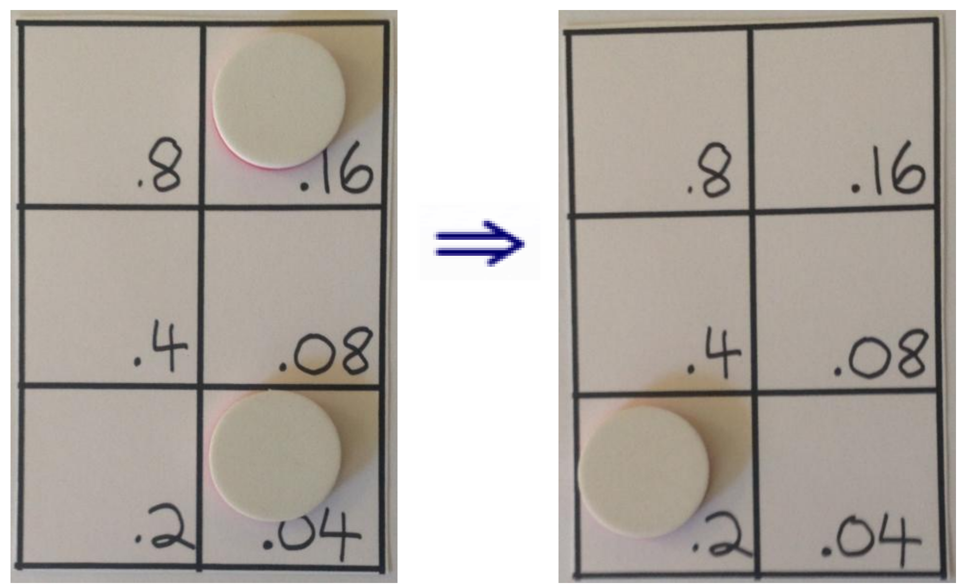


because he/she knows that 4 + 1 = 5.

But the student can “discover” that
.16 + .04 = .2,
because it is obtained by the same “move” on the board:



On a fractional part of a decimal board the move looks like this:



Using boards with college students, teachers, and middle school students

We have already tested some ways of teaching arithmetic using counting boards in several college math courses attended by preservice elementary and middle school teachers and by students interested in teaching mathematics on the secondary or college level.

Some materials have been used by teachers in their own classrooms, and during “enrichment” activities at the university for middle school students from local schools.

The results are encouraging. Most participants find activities with the boards easy to learn, but interesting and challenging.

The tasks we used belonged to two categories.

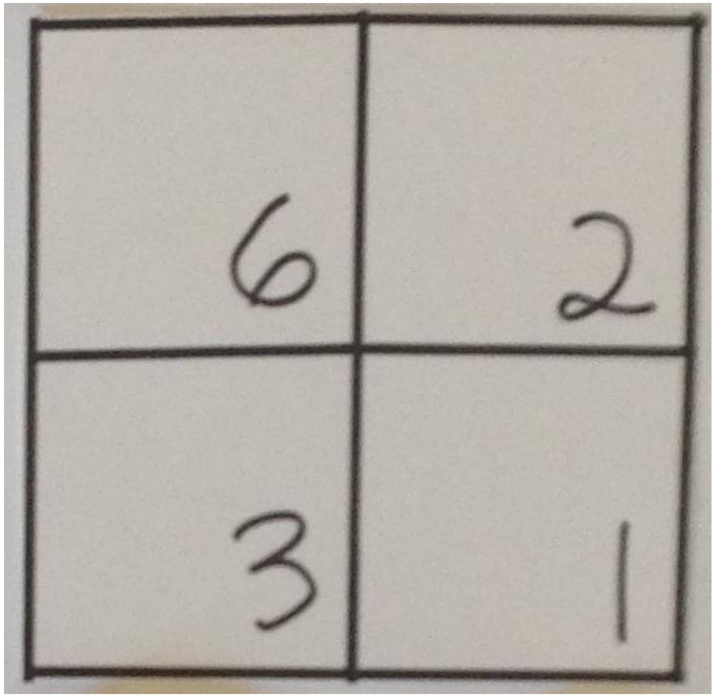
1. Students were shown how to carry out algorithms for addition, subtraction, multiplication and division for positive and negative decimals, as an alternative to computation with paper and pencil.

These activities were interesting only to some students; a typical comment: "I already know one way of doing it, and I am not interested in learning another way."

2. Students were given problems dealing with representing numbers on a board, and they had to figure out their own ways to solve them.

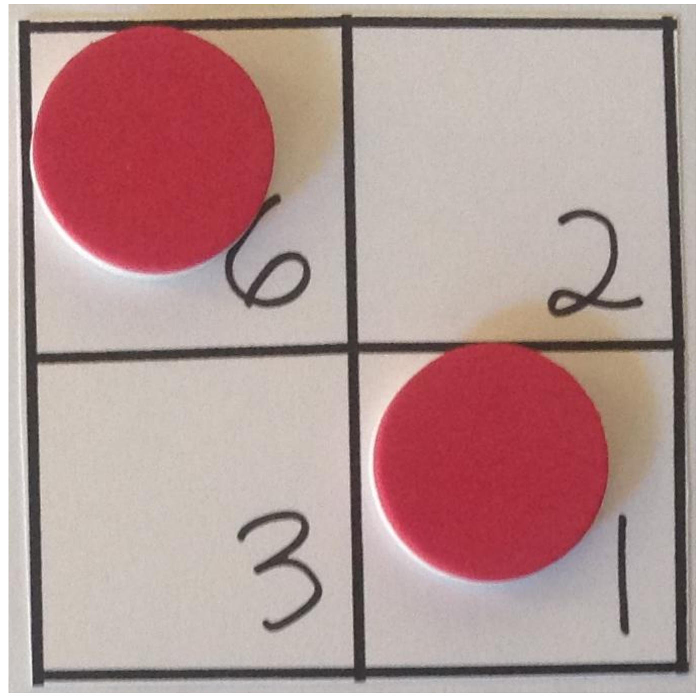
These problems were interesting and challenging to the large majority of students, and we got very positive feedback about the tasks.

Example 1



You may use both white (positive) and red (negative) tokens. But you may put at most one token on each square. What numbers can you put on on this board? Can you put -7?

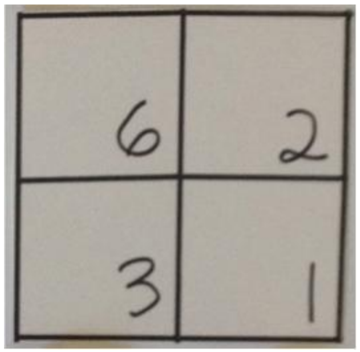
One way to put -7 on the board:



Can you put 13 on the board?

Not if you are limited to at most one token per square.

In how many ways can you make zero on the 2 by 2 board?



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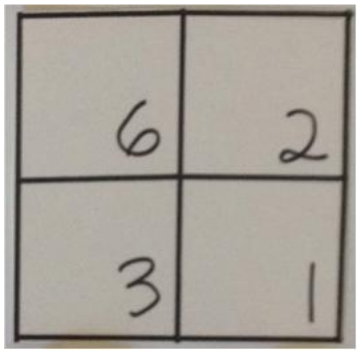
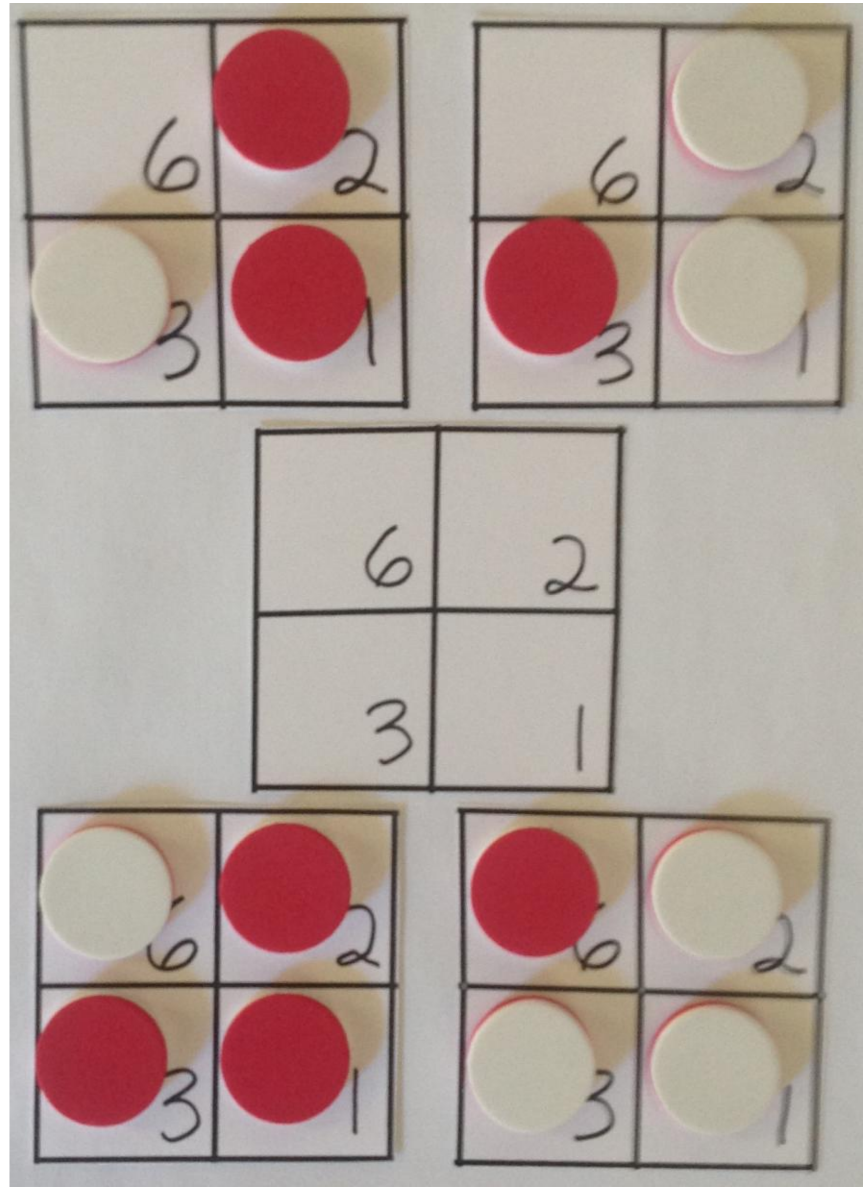
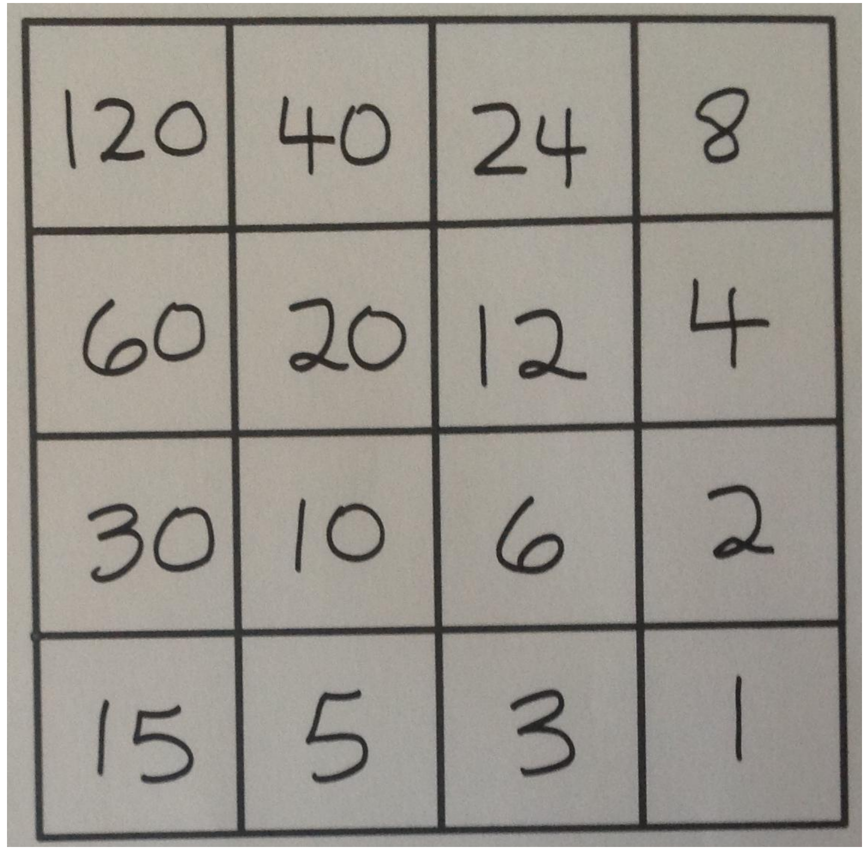


There are 5 ways:

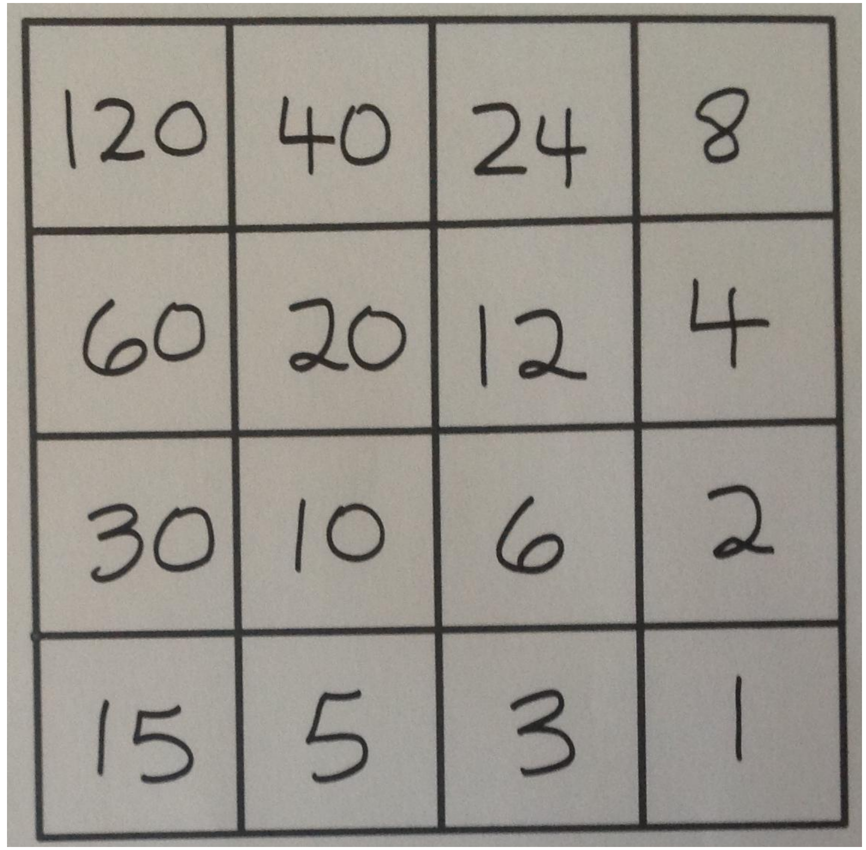


Example 2



With at most one token per square, what is the biggest number you can put on this board?

Example 2



With at most one token per square, what is the biggest number you can put on this board?

Answer: 360

Example 3 (a class project)

120	40	24	8
60	20	12	4
30	10	6	2
15	5	3	1

How to put the numbers 1 through 61 on the counting board with exactly two tokens? You may include more than one solution.

Disclaimer

There are many possible reasons to teach arithmetic. For example,

- -practical reasons
- -social reasons
- -cultural reasons
- -intellectual reasons

- -teach arithmetic for skills, for understanding, …

We are not getting involved in this discussion.
But we think that using counting boards can be a good teaching tool.

5. Interesting mathematical problems

Many problems concerning properties of counting boards are difficult and challenging.

Two examples

1. Consider boards of any size containing only unit fractions,

1	1/3	1/5	1/7	…
½	1/6	1/10	1/14	…
¼	1/12	1/20	1/28	…
…	…	…	…	…

You may use only tokens with value 1 (no negative numbers!). And you may put only one token at any location.

Question
Can any proper fraction with numerator 4 be represented by three or fewer tokens?
(A positive answer to this question is the Erdös-Straus Conjecture which was formulated in 1948 and is still unproven.)

2. Consider decimal boards of any size. You may use negative numbers, using tokens with values 1 and -1.

…	…	…	…	…
…	10	2	.4	…
…	5	1	.2	…
…	2.5	.5	.1	…
…	…	…	…	…

In how many different ways can you represent the number 3 with two tokens?

This is a nice problem for middle school students, and the eight answers are

2 + 1	4 + -1	5 + -2	8 + -5
128 + -125	2.5 + .5	3.2 + -.2	3.125 + -.125

But are these the only answers? We don't know.
(We only know that this problem is decidable.)

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But are these the only answers? We don't know.
(We only know that this problem is decidable.)
If you can prove that there are only eight solutions, please let us know!

Thank you!

Comments? Questions? Suggestions?

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