University students learn a new long division algorithm

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Patricia Baggett
Dept. of Math Sciences
New Mexico State University
Las Cruces, NM 88003-8001
baggett@nmsu.edu

Andrzej Ehrenfeucht
Computer Science Dept.
University of Colorado
Boulder, CO 80309-0430
andrzej.ehrenfeucht@colorado.edu

http://www.math.nmsu.edu/~breakingaway

Outline

1. Introduction
2. Algorithm for division
3. Data
   - Participants
   - Auxiliary algorithms
   - Results
4. Conclusions

Introduction

In order to master the written algorithms for addition, subtraction, multiplication, and division that are taught in schools, children have to memorize addition facts up to 9+9 and multiplication facts up to 9*9.

Besides this, these algorithms applied to multi-digit numbers put a strain on a student's working memory, requiring repeated use of the result of one calculation in the next one. This problem is alleviated by many ad hoc methods that are unreliable and error-prone.

Why just these particular algorithms for the four arithmetic operations have become a staple of general education in mathematics has a good historical explanation (we will not discuss it here).
But the choice is not justified either by their mathematical merits or by their educational or psychological merits.

From a practically infinite variety of arithmetic algorithms, we selected an analyzed four (one for each arithmetic operation), which have the following properties:

(1) To carry them out, students need to memorize just addition and subtraction facts, and only for numbers with sums up to 10. (No multiplication facts are needed.)

(2) The algorithms are modular, so using previously computed values in the operations that come next is very limited, even for multi-digit numbers.

(3) They are as efficient, and almost as fast, as the standard algorithms.

In this talk we present an algorithm for division, and we show data regarding how it was learned by a group of undergraduate students in a mathematics class for future elementary and middle school teachers.

2. Algorithm for division

(shown via an example)

Task: Compute $37\overline{)2556}$

(1) Compute multiples of 37 by 2, 3, and 6, by *doubling* and *adding*:

$$37\overline{)2556}$$

double 37	74
add 37 and 74	111
double 111	222

(2) Subtract precomputed multiples from the multiplicand, as in the normal algorithm, but without copying them (mentally subtract the numbers which are not aligned).

```
        6
 37 ) 2556
 74     33      ← 255 - 222 = 33
111
222
```

```
         66
    37)2556
    74    336      ← bring down 6
   111   114       ←  336 - 222 = 114
   222
```

	3	quotient = 69, remainder = 3
	66	
37)	2556	
74	336	
111	114	
222	3	← 114 − 111 = 3

Copying numbers for subtraction would lead to the computation on the right, which is slower than the one on the left because there is more writing.

$$
\begin{array}{rr}
 & 3 \\
 & 66 \\
37) & \overline{2556} \\
74 & 336 \\
111 & 114 \\
222 & 3
\end{array}
\qquad\qquad
\begin{array}{rr}
 & 3 \\
 & 66 \\
37) & \overline{2556} \\
74 & \underline{222} \\
111 & 336 \\
222 & \underline{\underline{222}} \\
 & 114 \\
 & \underline{\underline{111}} \\
 & 3
\end{array}
$$

Data

Participants

Twenty-four students in an undergraduate math class for future teachers were taught the new algorithm. Their participation was voluntary and did not count toward their grade. The four quizzes were anonymous, signed only by a student's chosen code.

Auxiliary algorithms

Doubling

Students were shown how to *double* numbers efficiently, and how to add and subtract without getting over 10. But they were allowed to carry out these operations any way they wanted.

Example of doubling 8 2 4 7

8 2 4 7

6 4 4 If the digit to the right is < 5, record the ones digit in the double of the digit.

1 6 4 9 4 If the digit to the right is ≥ 5, record the ones digit in the double of the digit, plus one. The left-most digit in the number to be doubled is zero; don’t forget it!

Subtracting without getting over ten

Example

$45 - 7$

Instead of thinking, $30 + 15 - 7 = 30 + 8$,
you may think, $40 - 10 + 5 + 3$, because $7 = 10 - 3$.

Some students wrote regrouping above the numbers during subtraction; for example,

$$\begin{array}{rrr} 2 & 3 & 5 \\ - & 8 & 6 \\ \hline \end{array}$$

is computed as follows:

$$\begin{array}{rrr} 1 & 12 & 15 \\ 2 & 3 & 5 \\ - & 8 & 6 \\ \hline 1 & 4 & 9 \end{array}$$

They could not do this in this division algorithm. But they were shown how to *mark* places between the digits where regrouping is needed (hence the name "marked" division). This is a quite adequate replacement for written regrouping, even when you subtract two digits, one of which is not written.

Marking

An example

$$\begin{array}{ccc} 2_{|} & 3_{|} & 5 \\ - & 8 & 6 \\ \hline 1 & 4 & 9 \end{array}$$

Students practiced subtraction (without copying the number), doubling, and division, during 16 sessions, approximately 10 minutes long. They were given four assessments (2 division problems) on four different days.

Results

We report students’ scores on four quizzes, each consisting of two problems.

Students’ participation in the four quizzes

	in all 4	in 3	in 2	in 1
No. of students	8	5	6	5

Problem	Answer	No. students w/ all correct	No. w/ one wrong digit	>1 error or no answer
quiz 1 17)1116	65 r 11	9 (56%)	3 (19%)	4 (25%)
356)13176	37 r 4	9 (56%)	0	7 (44%)
quiz 2 43)27838	647 r 17	11 (58%)	4 (21%)	4 (21%)
651)235764	362 r 102	10 (53%)	3 (16%)	6 (32%)
quiz 3 27)2629	97 r 10	10 (71%)	2 (14%)	2 (14%)
467)28962	62 r 8	10 (71%)	4 (28%)	0
quiz 4 87)43821	503 r 60	10 (63%)	5 (31%)	1 (6%)
562)352647	627 r 273	9 (56%)	4 (25%)	3 (19%)
On all problems		78 (60%)	25 (19%)	27 (21%)

During the first assessment, students were asked to answer the following questions:

1. Could you explain to someone else how to do marked division? (Yes, I'm not sure, No)
2. Is marked division simpler or more complex than the method you used before? (simpler, about the same, more complex)

Results

Question 1			Question 2		
Yes	Not sure	No	simpler	same	more complex
7 (47%)	6 (40%)	2 (13%)	2 (14%)	4 (29%)	8 (57%)

During the third assessment students were asked to answer the following two questions:

1. If you think that the marked division method is more difficult than the method you learned before, explain why you think it is more difficult.
If you think marked division is easier than the method you learned before, explain why you think it is easier.

2. Do you think that school children should be shown this method? (yes, no, no answer). Why or why not?

Question 1
(compare to Q. 2 above)

simpler	same	more complex
2 (14%)	2 (14%)	10 (72%)

(earlier answers were

2 (14%)	4 (29%)	8 (57%))

Question 2 (teach to children?)

Yes	no answer	No
10 (76%)	2 (15%)	1 (8%)

The apparent discrepancy between students' perception of the difficulty of the new algorithm and their recommending it for school use becomes clear when we compare reasons for each of them. (A and B below)

A. Reasons why the new algorithm is more difficult (10 answers):

I am more familiar with the old one. (6 answers):

“I think it is harder because it is difficult for me to switch the way I do it.”

“Because I was used to the other method…”

“…since I am not used to it…”

“…it is harder after so many years of knowing division another way.”

Subtraction without copying the numbers is difficult (2 answers):

“…I don’t like not being able to write out the subtraction. That makes it harder.”

“…I have to see visually the numbers being subtracted.”

Other reasons (2 answers):

“Because there are steps to remember.”

Why the algorithms are of the same difficulty:

> "Because they both can be confusing depending on the problem."

Why the new algorithm is easier:

> "There is no multiplication." (one answer)

B. Why to teach it to children (10 answers)

Because it is easier, simpler (6 answers):

"It is kind of simple, it will help them a lot."

"Definitely!"

"Those who don't get it the traditional way
May get this method right away."

"…because it may be easier for them…"

"It is easier for children to count [?] than to multiply."

Because it is good to learn different methods (3 answers):

"…it would be helpful to practice different operations to get the same answer."

"…it gives them a different alternative."

No reason (one)

Why not to teach it to children (one answer):

"…too complicated"

Conclusions

A specific question we wanted to answer in this study was: Will subtraction without copying the number, which speeds up the process, be perceived as the main difficulty?
The answer is no. (Only two of 14 students listed it as such.)

Most students learned the algorithm, although the rate of errors remained high. (We think that it is comparable to the rate of errors when the standard algorithm is used on problems of the same complexity, but we did not test this.)

When asked to evaluate the difficulty of the new algorithm, most students chose a “personal” point of view: “It was difficult for me because…”, and they did not try to compare technical aspects of both algorithms, such as multiples (2, 3, and 6) versus using multiplication facts.

The same students (with a personal point of view) thought that the algorithm would be easier for children to learn, but they did not suggest that it should *replace* the current algorithm.

Both the long multiplication and the long division algorithm currently taught in schools require mastery of multiplication facts up to 9*9, and they put a strain on a student's working memory by requiring that the tens-digit of the previous product be added to the next fact recalled.

"Marked" division is an example of an algorithm that avoids both of these difficulties.